Exercises on "the Growth of Functions"

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Introduction

Last week, we have learned two different ways to decide whether one function $f(n)$ grows faster than another $g(n)$:

- The first one achieves the purpose by finding appropriate "constants c_1, c_2".
- The second is by inspecting the ratio $\frac{f(n)}{g(n)}$ as $n \to \infty$.

In this tutorial, we will apply both methods through some exercises.

Exercise 1

Let $f(n) = 10n + 5$ and $g(n) = n^2$. Prove $f(n) = O(g(n))$ and $g(n) \neq O(f(n))$.

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Direction 1: Constant Finding

$f(n) = O(g(n))$, if there exist two constants c_1 and c_2 such that $f(n) \leq c_1 \cdot g(n)$ holds for all $n \geq c_2$.

Exercise 1

Let $f(n) = 10n + 5$ and $g(n) = n^2$. Prove $f(n) = O(g(n))$ and $g(n) \neq O(f(n))$.

Direction 1: Constant Finding

Proof of $f(n) = O(g(n))$

Our mission is to find c_1, c_2 to make $f(n) \leq c_1 \cdot g(n)$ hold for all $n \geq c_2$. Remember: we do not need to find the smallest c_1, c_2; instead, it suffices to obtain any c_1, c_2 that can do the job. Indeed, we will often go for some "easy" selections that can simplify derivation.

Exercise 1

Let $f(n) = 10n + 5$ and $g(n) = n^2$. Prove $f(n) = O(g(n))$ and $g(n) \neq O(f(n))$.

Direction 1: Constant Finding

Proof of $f(n) = O(g(n))$

Setting $c_1 = 10$, we want:

$$\begin{aligned} & 10n + 5 \leq 10 \cdot n^2 \\ \Leftrightarrow \quad & 5 \leq 10n(n-1) \\ \Leftarrow \quad & 5 \leq 10n \qquad (\text{for } n \geq 2) \\ \Leftrightarrow \quad & 1/2 \leq n \end{aligned}$$

Hence, it suffices to set $c_2 = 2$.

Exercise 1

Let $f(n) = 10n + 5$ and $g(n) = n^2$. Prove $f(n) = O(g(n))$ and $g(n) \neq O(f(n))$.

Direction 1: Constant Finding

Proof of $g(n) \neq O(f(n))$

Let us prove this by contradiction. Suppose, on the contrary, that $g(n) = O(f(n))$. This means the existence of constants c_1, c_2 such that, we have for all $n \geq c_2$

$$
\begin{aligned}
& n^2 \leq c_1 \cdot (10n + 5) \\
\Rightarrow \quad & n^2 \leq c_1 \cdot 20n \\
\Leftrightarrow \quad & n \leq 20c_1
\end{aligned}
$$

which cannot always hold for all $n \geq c_2$. This completes the proof.

Exercise 1

Let $f(n) = 10n + 5$ and $g(n) = n^2$. Prove $f(n) = O(g(n))$ and $g(n) \neq O(f(n))$.

Direction 2: Inspecting $\lim_{n\to\infty} \frac{f(n)}{g(n)}$

Proof of $f(n) = O(g(n))$

$$\lim_{n\to\infty} \frac{10n+5}{n^2} = \lim_{n\to\infty} \frac{10+5/n}{n} = 0.$$

Hence, $f(n) = O(g(n))$.

Exercise 1

Let $f(n) = 10n + 5$ and $g(n) = n^2$. Prove $f(n) = O(g(n))$ and $g(n) \neq O(f(n))$.

Direction 2: Inspecting $\lim_{n\to\infty} \frac{f(n)}{g(n)}$

Proof of $g(n) \neq O(f(n))$

$$\lim_{n\to\infty} \frac{n^2}{10n+5} = \infty.$$

Hence, $g(n) \neq O(f(n))$.

Exercise 2

Let $f(n) = 5\log_2 n$ and $g(n) = \sqrt{n}$. Prove $f(n) = O(g(n))$ and $g(n) \neq O(f(n))$.

Direction 1: Constant Finding

Proof of $f(n) = O(g(n))$

Setting $c_1 = 5$, we want:

$$\begin{aligned} & 5\log_2 n \le 5 \cdot \sqrt{n} \\ \Leftrightarrow \quad & \log_2 n \le \sqrt{n} \end{aligned}$$

Hence, it suffices to set $c_2 = 64$.

Direction 1: Constant Finding

Proof of $g(n) \neq O(f(n))$

We prove this by contradiction. Suppose that $g(n) = O(f(n))$. It implies that there exist constants c_1, c_2 such that for all $n \geq c_2$, we have

$$\sqrt{n} \leq c_1 \cdot 5 \cdot \log_2 n$$
$$\Leftrightarrow \quad \frac{\sqrt{n}}{\log_2 n} \leq 5c_1$$

which cannot always hold for all $n \geq c_2$. This completes the proof.

Direction 2: Inspecting $\lim_{n \to \infty} \frac{f(n)}{g(n)}$

Proof of $f(n) = O(g(n))$

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \lim_{n \to \infty} \frac{5 \log_2 n}{\sqrt{n}} = 0.$$

Thus, we have $f(n) = O(g(n))$.

Proof of $g(n) \neq O(f(n))$.

$$\lim_{n \to \infty} \frac{g(n)}{f(n)} = \lim_{n \to \infty} \frac{\sqrt{n}}{5 \log_2 n} = \infty.$$

Hence, $g(n) \neq O(f(n))$.

Exercise 3

Given that $10n + 5 = O(n^2)$ and $5 \log_2 n = O(\sqrt{n})$, prove $10n + 5 + 5 \log_2 n = O(n^2 + \sqrt{n})$.

Direction 1: Constant Finding

Since $10n + 5 = O(n^2)$ implies the existence of constants c_1 and c_2 such that $10n + 5 \leq c_1 \cdot n^2$ holds for all $n \geq c_2$.

Similarly, $5 \log_2 n = O(\sqrt{n})$ means there exist two constants c_1' and c_2' which make $5 \log_2 n \leq c_1' \cdot \sqrt{n}$ hold for all $n \geq c_2'$.

Thus:

$$10n + 5 + 5 \log_2 n \leq c_1 n^2 + c_1' \sqrt{n} \leq \max\{c_1, c_1'\} \cdot (n^2 + \sqrt{n})$$

holds for all $n \geq \max\{c_2, c_2'\}$.

Therefore, $10n + 5 + 5 \log_2 n = O(n^2 + \sqrt{n})$.

Direction 2: Inspecting $\lim_{n\to\infty} \frac{f(n)}{g(n)}$

Since $10n + 5 = O(n^2)$, we have $\lim_{n\to\infty} \frac{10n+5}{n^2} = c$, where c is some constant.

Similarly, $5\log_2 n = O(\sqrt{n})$ indicates that $\lim_{n\to\infty} \frac{5\log_2 n}{\sqrt{n}} = c'$, where c' is some constant.

Both of the above imply that:

$$\begin{aligned}\lim_{n\to\infty} \frac{10n + 5 + 5\log_2 n}{n^2 + \sqrt{n}} &= \lim_{n\to\infty} \frac{10n+5}{n^2+\sqrt{n}} + \lim_{n\to\infty} \frac{5\log_2 n}{n^2+\sqrt{n}} \\ &\le \lim_{n\to\infty} \frac{10n+5}{n^2} + \lim_{n\to\infty} \frac{5\log_2 n}{\sqrt{n}} \\ &= c + c'.\end{aligned}$$

Therefore, $10n + 5 + 5\log_2 n = O(n^2 + \sqrt{n})$.

Exercise 4

Consider functions of n: $f_1(n)$, $f_2(n)$, $g_1(n)$ and $g_2(n)$ such that:

$$f_1(n) = O(g_1(n)) \text{ and } f_2(n) = O(g_2(n))$$

Prove $f_1(n) + f_2(n) = O(g_1(n) + g_2(n))$.

Direction 1: Constant Finding

Since $f_1(n) = O(g_1(n))$, there exist constants c_1 and c_2 such that $f_1(n) \leq c_1 \cdot g_1(n)$ holds for all $n \geq c_2$.

Similarly, $f_2(n) = O(g_2(n))$ implies the existence of constants c_1' and c_2' such that $f_2(n) \leq c_1' \cdot g_2(n)$ holds for all $n \geq c_2'$.

Thus:

$$f_1(n) + f_2(n) \leq c_1 \cdot g_1(n) + c_1' \cdot g_2(n) \leq \max\{c_1, c_1'\} \cdot (g_1(n) + g_2(n))$$

for all $n \geq \max\{c_2, c_2'\}$.

Therefore, $f_1(n) + f_2(n) = O(g_1(n) + g_2(n))$.

Direction 2: Inspecting $\lim_{n\to\infty} \frac{f(n)}{g(n)}$

Since $f_1(n) = O(g_1(n))$, we have $\lim_{n\to\infty} \frac{f_1(n)}{g_1(n)} = c$ for some constant c.

Similarly, $f_2(n) = O(g_2(n))$ indicates $\lim_{n\to\infty} \frac{f_2(n)}{g_2(n)} = c'$ for some constant c'.

This leads to:

$$\begin{aligned}
\lim_{n\to\infty} \frac{f_1(n) + f_2(n)}{g_1(n) + g_2(n)} &= \lim_{n\to\infty} \frac{f_1(n)}{g_1(n) + g_2(n)} + \lim_{n\to\infty} \frac{f_2(n)}{g_1(n) + g_2(n)} \\
&\leq \lim_{n\to\infty} \frac{f_1(n)}{g_1(n)} + \lim_{n\to\infty} \frac{f_2(n)}{g_2(n)} \\
&\leq c + c'.
\end{aligned}$$

Therefore, $f_1(n) + f_2(n) = O(g_1(n) + g_2(n))$.