Priority Queues and Binary Heaps

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In this lecture, we will learn our first "tree data structure"—called the binary heap—which serves as an implementation of the priority queue.

Priority Queue

A priority queue stores a set S of n integers and supports the following operations:

- `Insert`(e): Adds a new integer to S.
- `Delete-min`: Removes the smallest integer in S, and returns it.

Example

Suppose that the following integers are inserted into an initially empty priority queue: 93, 39, 1, 26, 8, 23, 79, 54.

If next we perform a `Delete-Min`, the operation returns 1, after which $S = \{93, 39, 26, 8, 23, 79, 54\}$.

The next `Delete-Min` returns 8, and leaves $S = \{93, 39, 26, 23, 79, 54\}$.

> Unlike an ordinary queue (which obeys FIFO), a priority queue guarantees that the elements always leave in ascending order, regardless of the order by which they are inserted.

Next we will implement a priority queue using a data structure called the binary heap to achieve the following guarantees:

- $O(n)$ space consumption
- $O(\log n)$ insertion time
- $O(\log n)$ delete-min time.

Binary Heap

Let S be a set of n integers. A binary heap on S is a binary tree T satisfying:

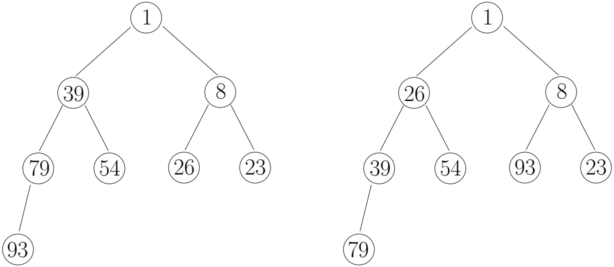
1. T is complete.
2. Every node u in T corresponds to a distinct integer in S—the integer is called the key of u (and is stored at u).
3. If u is an internal node, the key of u is smaller than those of its child nodes.

Note:

- Condition 2 implies that T has n nodes.
- Condition 3 implies that the key of u is the smallest in the subtree of u.

Example

Two possible binary heaps on $S = \{93, 39, 1, 26, 8, 23, 79, 54\}$:



The smallest integer of S must be the key of the root.

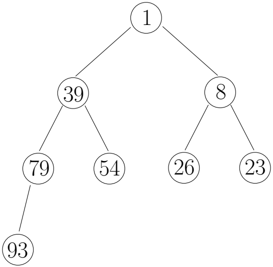
Insertion

We perform insert(e) on a binary heap T as follows:

1. Create a leaf node z with key e, while ensuring that T is a complete binary tree—notice that there is only one place where z can be added.
2. Set $u \leftarrow z$.
3. If u is the root, return.
4. If the key of $u >$ the key of its parent p, return.
5. Otherwise, swap the keys of u and p. Set $u \leftarrow p$, and repeat from Step 3.

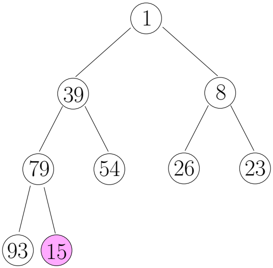
Example

Assume that we want to insert 15 into the binary heap below:



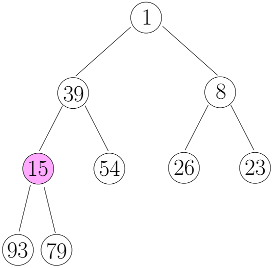
Example

First, add 15 as a new leaf, making sure that we still have a complete binary tree.



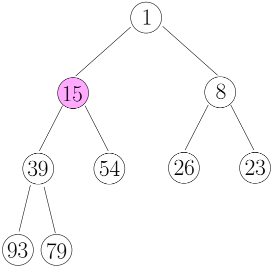
15 causes a violation by being smaller than its parent. This is fixed by a swap with its parent; see next.

Example



15 still causes a violation, necessitating another swap, as shown next.

Example



No more violation. Insertion complete.

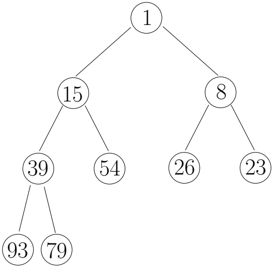
Delete-Min

We perform a delete-min on a binary heap T as follows:

1. Report the key of the root.
2. Identify the rightmost leaf z at the bottom level of T.
3. Delete z, and store the key of z at the root.
4. Set $u \leftarrow$ the root.
5. If u is a leaf, return.
6. If the key of $u <$ the keys of the children of u, return.
7. Otherwise, let v be the child of u with a smaller key. Swap the keys of u and v. Set $u \leftarrow v$, and repeat from Step 5.

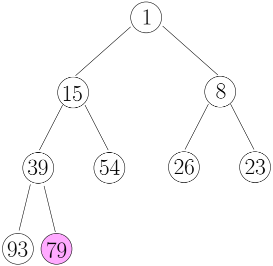
Example

Assume that we perform a delete-min from the binary heap below:



Example

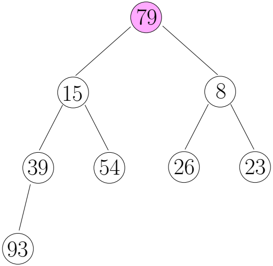
First, find the rightmost leaf at the bottom level, namely, 79.



Notice that the tree is still a complete binary tree after removing this leaf.

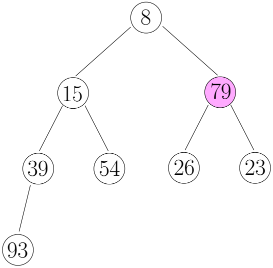
Example

Remove the leaf, but place the value 79 in the root.



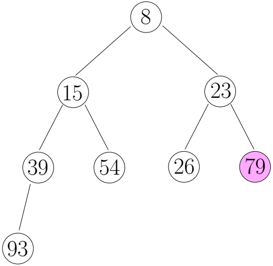
79 causes a violation by being greater than its children. This is fixed by swapping it with node 8, which is the child of the root with a smaller key. See the next slide.

Example



Node 79 still has a violation, causing another swap as shown next.

Example



The final tree after the delete-min.

How to Find the Rightmost Leaf at the Bottom Level

Before analyzing the running time of `insert` and `delete-min`, let us first consider a sub-problem:

> Given a complete binary tree T with n nodes, how to identify quickly the rightmost leaf node at the bottom level of T.

Our aforementioned algorithms depend on a fast solution to the above.

How to Find the Rightmost Leaf at the Bottom Level

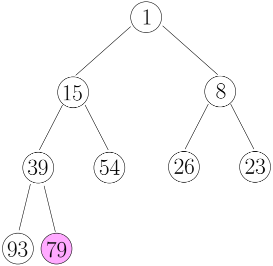
Next, we give a clever algorithm for solving the sub-problem in $O(\log n)$ time.

First, write the value of n in binary form.

Skip the most significant bit. We will scan the remaining bits from left to right, and descend as instructed by the next bit:

- If the next bit is 0, we go to the left child of the current node.
- Otherwise, go to the right child.

Example



Here $n = 9$, which is 1001 in binary. Skipping the first bit 1, we scan the remaining bits and descend accordingly:

- The 2nd leftmost bit is 0; so we turn left, and go to node 15.
- The 3rd leftmost bit is 0; so we turn left, and go to node 39.
- The 4th leftmost bit is 1; so we turn right, and go to node 79 (done).

Analysis of Insertion and Delete-Min

We are now ready to prove that our `insertion` and `delete-Min` algorithms finish in $O(\log n)$ time.

It suffices to point out the key facts:

- Step 1 of the insertion algorithm (Slide 8) and Step 2 of the delete-min algorithm (Slide 13) can be performed in $O(\log n)$ time, using our solution to the previous sub-problem.
- The rest of insertion ascends a root-to-leaf path, while that of delete-min descends a root-to-leaf path. The time is $O(\log n)$ in both cases.

Now officially we have reached the following conclusion. We can maintain a priority queue on a set S of elements such that:

- At any moment, the space consumption is $O(n)$, where $n = |S|$.
- An insertion can be processed in $O(\log n)$ time.
- A delete-min can be processed in $O(\log n)$ time.