Hashing

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In this lecture, we will revisit the dictionary search problem, where we want to locate an integer v in a set of size n or declare the absence of v. Recall that binary search solves the problem in $O(\log n)$ time. We will bring down the cost to $O(1)$ in expectation.

Towards the purpose, we will learn our first randomized data structure in this course. The structure is called the hash table.

The Dictionary Search Problem (Redefined)

S is a set of n integers. We want to preprocess S into a data structure so that queries of the following form can be answered efficiently:

- Given a value v, a query asks whether $v \in S$.

We will measure the performance of the data structure by examining its:

- Space consumption: How many memory cells occupied.
- Query cost: Time of answering a query.
- Preprocessing cost: Time of building the data structure.

Dictionary Search—Solution Based on Binary Search

We can solve the problem by sorting S into an array of length n, and using binary search to answer a query. This achieves:

- Space consumption: $O(n)$.
- Query cost: $O(\log n)$.
- Preprocessing cost: $O(n \log n)$.

Dictionary Search—This Lecture (the Hash Table)

We will improve the previous solution in expectation:

- Space consumption: $O(n)$.
- Query cost: $O(\log n) \Rightarrow O(1)$ in expectation.
- Preprocessing cost: $O(n \log n) \Rightarrow O(n)$.

Hashing

The main idea of hashing is to divide the dataset S into a number m of disjoint subsets such that:

- only one subset needs to be searched to answer any query.

Hash Function

Let $\mathbb{Z}$ denote the set of all integers, and $[m]$ the set of integers from 1 to m.

A *hash function* h is a function from $\mathbb{Z}$ to $[m]$. Namely, given any integer k, $h(k)$ returns an integer in $[m]$.

The value $h(k)$ is called the *hash value* of k.

Any hash function produces a hash table that correctly solves the dictionary search problem. However, the quality of the function has a heavy impact on the query efficiency.

Hash Table – Preprocessing

First, choose an integer $m > 0$, and a hash function h from $\mathbb{Z}$ to $[m]$.

Then, preprocess the input S as follows:

1. Create an array H of length m.
2. For each $i \in [1, m]$, create an empty linked list L_i. Keep the head and tail pointers of L_i in $H[i]$.
3. For each integer $x \in S$:
   - Calculate the hash value $h(x)$.
   - Insert x into $L_{h(x)}$.

Space consumption: $O(n + m)$.
Preprocessing time: $O(n + m)$.

We will always choose $m = O(n)$, so $O(n + m) = O(n)$.

Hash Table – Querying

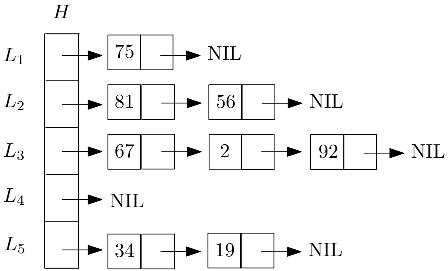
We answer a query with value v as follows:

1. Calculate the hash value $h(v)$.
2. Scan the whole $L_{h(v)}$. If v is not found, answer "no"; otherwise, answer "yes".

Query time: $O(|L_{h(v)}|)$, where $|L_{h(v)}|$ is the number of elements in $L_{h(v)}$.

Example

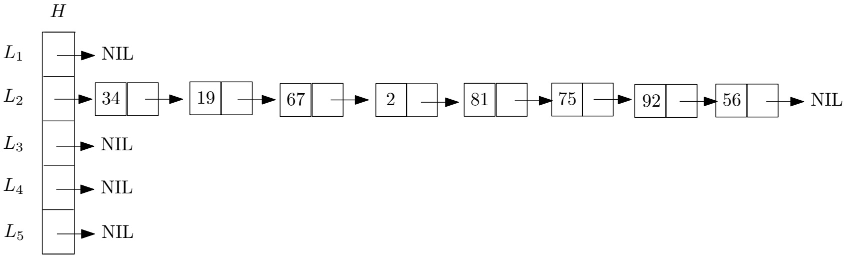
Let $S = \{34, 19, 67, 2, 81, 75, 92, 56\}$. Suppose that we choose $m = 5$, and $h(k) = 1 + (k \mod m)$.



To answer a query with search value 68, we scan all the elements in L_3, and answer "no". For this hash function, the maximum query time is the cost of scanning a linked list of 3 elements.

Example

Let $S = \{34, 19, 67, 2, 81, 75, 92, 56\}$. Suppose that we choose $m = 5$, and $h(k) = 2$.



For this hash function, the maximum query time is the cost of scanning a linked list of 8 elements (i.e., the worst possible).

It is clear that a good hash function should create linked lists of roughly the same size, i.e., "spreading out" the elements of S as evenly as possible.

In order to achieve $O(1)$ expected query time, we require that the hash function h (from $\mathbb{Z}$ to $[m]$) should be chosen from a large family of functions to ensure the following universal property:

The following holds for any two different integers k_1, k_2:

$$\boldsymbol{Pr}[h(k_1) = h(k_2)] \quad \leq \quad \frac{1}{m}$$

Next, we will first prove that universality gives us the desired $O(1)$ expected query time. Then, we will describe a way to obtain such a good hash function.

Analysis of Query Time under Universality

We focus on the case where q does not exist in S (the case where it does is similar). Recall that our algorithm probes all the elements in the linked list $L_{h(q)}$. The query cost is therefore $O(|L_{h(q)}|)$.

Define random variable X_i ($i \in [1, n]$) to be 1 if the i-th element e of S has the same hash value as q (i.e., $h(e) = h(q)$), and 0 otherwise. Thus:

$$|L_{h(q)}| = \sum_{i=1}^{n} X_i$$

Analysis of Query Time under Universality

By universality, $\textbf{\textit{Pr}}[X_i = 1] \le 1/m$, meaning that

$$\begin{aligned} \textbf{\textit{E}}[X_i] &= 1 \cdot \textbf{\textit{Pr}}[X_i = 1] + 0 \cdot \textbf{\textit{Pr}}[X_i = 0] \\ &\le 1/m. \end{aligned}$$

Hence:

$$\textbf{\textit{E}}[|L_{h(q)}|] = \sum_{i=1}^{n} \textbf{\textit{E}}[X_i] \le n/m.$$

By choosing $m = \Theta(n)$, we have $n/m = \Theta(1)$.

Designing a Universal Function

- Pick a prime number p such that
  - $p \geq m$.
  - $p \geq$ any possible integer k (see the next slide)
- Choose a number α uniformly at random from $1, ..., p-1$.
- Choose a number β uniformly at random from $0, ..., p-1$.
- Construct a hash function:

$$h(k) = 1 + (((\alpha k + \beta) \mod p) \mod m)$$

The proof of universality is not required in this course, but will be covered in the training camp.

Designing a Universal Function

You may be wondering why it is even possible to guarantee:

- $p \geq$ any possible integer k.

In fact, in the RAM model, the set of "possible integers" is actually finite. Recall that the RAM model is defined with a word length w, namely, the number of bits in a word. Hence, the maximum integer is $2^w - 1$.

Number theory shows that there is at least one prime number between x and $2x$, for $n > 3$. Hence, p always exists in the range $[2^w, 2^{w+1}]$.

Now officially we have shown that, for any set S of n integers, it is always possible to construct a hash table with the following guarantees on the dictionary search problem:

- Space $O(n)$.
- Preprocessing time $O(n)$.
- Query time $O(1)$ in expectation.