COMP3506/7505: Regular Exercise Set 8

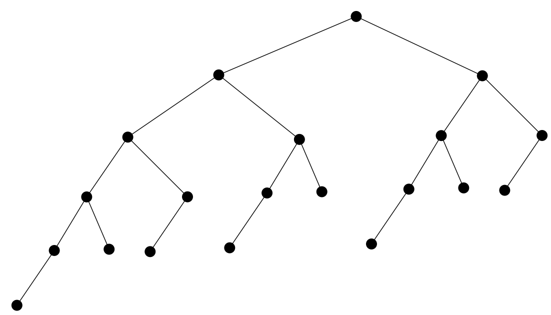
Prepared by Yufei Tao

Problem 1. In the class, we proved that if $f(h)$ denotes the smallest number of nodes in a balanced binary tree of height h, it must hold that

$$f(h) = 1 + f(h-1) + f(h-2).$$

Give a balanced binary tree of height 6 with $f(6)$ nodes.

Solution.



Problem 2. Let T be a binary tree of n nodes. For each node u of T, define its *count* as the number of nodes in its subtree (remember that the subtree includes the node itself). Describe an algorithm to compute the counts of all the nodes in T (you can assume that each node has reserved a memory cell for you to store the count). Your algorithm must terminate in $O(n)$ time.

Solution. We will design a recursive algorithm to perform a *post order* traversal of T. This algorithm at its termination will have computed the counts of all the nodes.

If the tree has only a single node, set its count to 1 and return. Otherwise, the algorithm proceeds as follows:

- Recursively compute the counts of all the nodes in the left subtree of the root.
- Recursively compute the counts of all the nodes in the right subtree of the root.
- Let c_1 be the count of the left child, and c_2 be the count of the right child. Set the count of the root to $1 + c_1 + c_2$.

To see that the running time is $O(n)$, observe that the algorithm essentially crosses each edge of the tree twice: once from the parent to the child, and another time the other way around. There are only $n - 1$ edges in the tree.

Problem 3. Let T be a binary search tree (BST) of on a set S of n integers. Let x and y be two integers in S. Describe an algorithm to find the lowest common ancestor A of the nodes in T storing x and y, respectively. If A is at level ℓ (recall that the root is at level 0), your algorithm must finish in $O(1 + \ell)$ time.

Solution. Without loss of generality, suppose that $x < y$. Let v be the root of T. If the key k of v equals either x or y, return v. Otherwise:

- If $x < k < y$, report v, and finish.

- If $y < k$, set v to its left child, and repeat the above steps at (the new) v.
- Otherwise, set v to its right child, and repeat the above steps at (the new) v.

Problem 4. Let T be a binary search tree (BST) of on a set S of n integers. Describe an $O(\log n + k)$-time algorithm to answer the following query: given an interval $[a, b]$, report all the integers of S that fall in $[a, b]$. Here, k is the number of integers reported.

Solution. First, find the successor a' of a, and the predecessor b' of b. Then, find the lowest common ancestor, denoted as node A, of the nodes a' and b'. This takes $O(\log n)$ time in total.

Denote by Π_1 the path from A to node a', and Π_2 the path from A to node b'. For every node on these two paths, report its key if the key falls in $[a, b]$. This takes $O(\log n)$ time.

For every node u on Π_1 other than A, do the following: if u is the left child of its parent p, then report all the keys in the right subtree of p. If k_u keys are reported, this step takes $O(1 + k_u)$ time.

For every node u on Π_2 other than A, do the following: if u is the right child of its parent p, then report all the keys in the left subtree of p. If k_u keys are reported, this step takes $O(1 + k_u)$ time.

Overall the cost is $O(\log n + k)$, noticing that

$$\sum_{u \in \Pi_1 \cup \Pi_2 \setminus \{A\}} k_u \leq k.$$

Problem 5. Let S be a set of n key-value pairs of the form (t, v). Denote by m the number of distinct keys in all the pairs of S. Describe a data structure to support the following queries efficiently: given an interval $[a, b]$, report all the pairs $(t, v) \in S$ such that $t \in [a, b]$. Your structure must use $O(n)$ space, and answer a query in $O(\log m + k)$ time, where k is the number of pairs reported.

Solution. Create a BST on the m distinct keys. At each node u of the tree, use a linked list to chain up all the key-value pairs (t, v) where t equals the key of u. The query algorithm is the same as the one for Problem 4, except that for every node u whose key falls in $[a, b]$, we should report everything in its linked list.