COMP3506/7505: Regular Exercise Set 10

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Problem 1. Let T be a (2, 3)-tree on a set S of integers. Suppose that each node in the tree stores a pointer to its parent. You are given the leftmost leaf z of T, and asked to report the k smallest integers in T. Describe an algorithm to do so in $O(k)$ time.

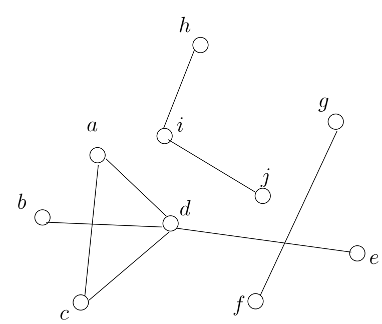
Problem 2. Let $G = (V, E)$ be a directed graph. Suppose that we perform BFS starting from a source vertex s, and obtain a BFS-tree T. For any vertex $v \in V$, denote by $l(v)$ the level of v in the BFS-tree. Prove that BFS en-queues the vertices v of V in non-descending order of $l(v)$.

Problem 3. Let $G = (V, E)$ be a directed graph. Suppose that we perform BFS starting from a source vertex s, and obtain a BFS-tree T. For any vertex $v \in V$, prove that the path from s to v in T is a shortest path from s to v in G.

Problem 4. Let $G = (V, E)$ be an undirected graph. We will denote an edge between vertices u, v as $\{u, v\}$. Next, we define the single source shortest path (SSSP) problem on G. Define a *path* from s to t as a sequence of edges $\{v_1, v_2\}, \{v_2, v_3\}, ..., \{v_t, v_{t+1}\}$, where $t \geq 1$, $v_1 = s$, and $v_{t+1} = t$. The *length* of the path equals t. Then, the SSSP problem gives a source vertex s, and asks to find shortest paths from s to all the other vertices in G. Adapt BFS to solve this problem in $O(|V| + |E|)$ time. Once again, you need to produce a BFS tree where, for each vertex $v \in V$, the path from the root to v gives a shortest path from s to v.

Problem 5 (Connected Components). Let $G = (V, E)$ be an undirected graph. A *connected component* (CC) of G includes a set $S \subseteq V$ of vertices such that

- For any vertices $u, v \in S$, there is a path from u to v, and a path from v to u.
- (Maximality) It is not possible to add any vertex into S while still ensuring the previous property.



For example, in the above graph, $\{a, b, c, d, e\}$ is a CC, but $\{a, b, c, d\}$ is not, and neither is $\{g, f, e\}$.

Prove: Let S_1, S_2 be two CCs. Then, they must be disjoint, i.e., $S_1 \cap S_2 = \emptyset$.

Problem 6. Let $G = (V, E)$ be an undirected graph. Describe an algorithm to divide V into a set of CCs. For example, in the example of Problem 5, your algorithm should return 3 CCs: $\{a, b, c, d, e, \}, \{g, f\}$, and $\{h, i, j\}$.

Problem 7. Recall that, in the DFS algorithm, after we have grown a DFS-tree, we may need to re-start from an arbitrary vertex that remains white. Multiple re-starts may be necessary throughout the algorithm. Describe how to find the re-starting white vertices efficiently so that the overall execution time of the algorithm is $O(|V| + |E|)$.