Problem 1 (30%). Given 2D points \( p = (p[1], p[2]) \) and \( q = (q[1], q[2]) \), define
\[
\]
Prove: \( K(p, q) \) is a kernel function. Specifically, you need to show a mapping function \( \phi : \mathbb{R}^2 \rightarrow \mathbb{R}^d \) for some integer \( d \) such that \( K(p, q) \) equals the dot product of \( \phi(p) \) and \( \phi(q) \).

Answer:
\[
\phi(x) = (\sqrt{3}, 2x[2], \sqrt{2}(x[1])^2, \sqrt{2}(x[1]x[2])^2, 2\sqrt{2}(x[1])^3(x[2])^2)
\]

Problem 2 (30%). Consider a training set \( P \) including the points below

\[
A(-3, 3) \quad B(-1, -3) \quad C(3, -1) \quad D(1, 4)
\]

where the two dots have label 1, the box has label 2, and the cross has label 3. We have a 3-class linear classifier defined by vectors \( w_1 = (-1, 3), w_2 = (3, 0), \) and \( w_3 = (0, -1) \) (note that this classifier separates \( P \)). Calculate the margin of the classifier.

Answer: Let \( W = \{w_1, w_2, w_3\} \).

\[
\text{margin}(A \mid W) = \min \left\{ \frac{w_1 \cdot A - w_2 \cdot A}{\sqrt{2 \times \sum_{i=1}^{3} |w_i|^2}}, \frac{w_1 \cdot A - w_3 \cdot A}{\sqrt{2 \times \sum_{i=1}^{3} |w_i|^2}} \right\} = \min \left\{ \frac{12 - (-9)}{\sqrt{40}}, \frac{12 - (-3)}{\sqrt{40}} \right\} = \frac{15}{\sqrt{40}}
\]

Similarly,

\[
\text{margin}(B \mid W) = \min \left\{ \frac{3 - (-8)}{\sqrt{40}}, \frac{3 - (-3)}{\sqrt{40}} \right\} = \frac{6}{\sqrt{40}}
\]

\[
\text{margin}(C \mid W) = \min \left\{ \frac{9 - (-6)}{\sqrt{40}}, \frac{9 - 1}{\sqrt{40}} \right\} = \frac{8}{\sqrt{40}}
\]

\[
\text{margin}(D \mid W) = \min \left\{ \frac{11 - 3}{\sqrt{40}}, \frac{11 - (-4)}{\sqrt{40}} \right\} = \frac{8}{\sqrt{40}}
\]
Therefore, the classifier’s margin equals $\frac{6}{\sqrt{40}}$.

**Problem 3 (40%).** Consider the set $P$ of points below:

(i) Run the $k$-center algorithm on $P$ under Euclidean distance. Suppose that $k = 3$ and the first center chosen is $a$. Explain the second and third centers found by the algorithm.

(ii) Run the $k$-means algorithm on $P$ with $k = 3$ under Euclidean distance, assuming that the algorithm selects a set $S = \{a, d, g\}$ as the initial centroids. Recall that the algorithm updates $S$ iteratively. Give the content of $S$ after each iteration until the algorithm terminates.

**Answer:**

(i) The second center is $g$ and the third is $h$.

(ii) *Iteration 1.* Let $o_1 = a, o_2 = d$, and $o_3 = g$. The algorithm divides $P$ into partitions $P_1$, $P_2$ and $P_3$ such that $P_i$ ($1 \leq i \leq 3$) includes all the points in $P$ with $o_i$ as their closest centroids. Specifically, $P_1 = \{a, b, c\}, P_2 = \{d, e, h, i, j\}$, and $P_3 = \{f, g\}$. Then, the algorithm resets $o_i$ to the geometric centroid of $P_i$: $o_1 = \left(\frac{8}{3}, 3\right), o_2 = (6, 7), \text{ and } o_3 = (9, 5)$.

*Iteration 2.* The algorithm re-divides $P$ into $P_1, P_2$ and $P_3$ based on the current centroids: $P_1 = \{a, b, c\}, P_2 = \{h, i, j\}$, and $P_3 = \{d, e, f, g\}$. Accordingly, the centroids are re-computed as $o_1 = \left(\frac{8}{3}, 3\right), o_2 = \left(\frac{16}{3}, \frac{26}{3}\right)$, and $o_3 = (8, \frac{19}{4})$.

*Iteration 3.* We get $P_1 = \{a, b, c\}, P_2 = \{h, i, j\}$, and $P_3 = \{d, e, f, g\}$ again after re-dividing $P$ based on the current centroids. The algorithm terminates.