CMSC5724: Exercise List 6

Problem 1. Prove the theorem on Slide 6 of the lecture notes on the kernel method without the interleaving assumption.

Problem 2. Consider the kernel function $K(p, q) = (\boldsymbol{p} \cdot \boldsymbol{q} + 1)^3$, where $\boldsymbol{p} = (p[1], p[2])$ and $\boldsymbol{q} = (q[1], q[2])$ are 2-dimensional vectors. Recall that there is a mapping function ϕ from $\mathbb{R}^2$ to $\mathbb{R}^d$ for some integer d such that $K(p, q)$ equals the dot product of $\phi(p)$ and $\phi(q)$. Give the details of ϕ.

Problem 3. Consider a set P of 2D points each labeled either -1 or 1. It is known that the points of the two labels can be linearly separated after applying the Kernel function $K(p, q) = (\boldsymbol{p} \cdot \boldsymbol{q} + 1)^2$. Prove that they can also be linearly separated by applying the kernel function $K'(p, q) = (2\boldsymbol{p} \cdot \boldsymbol{q} + 3)^2$.

Problem 4. Consider a set P of 2D points that has three label-1 points $p_1(-2, -2), p_2(1, 1), p_3(3, 3)$, and two label-$(-1)$ points $q_1(-2, 2), q_2(2, -2)$. Answer the following questions:

- Use Perceptron to find a separation plane π using the Kernel function $K(x, y) = (x \cdot y + 1)^2$.
- According to π, what is the label of point $(2, 2)$?

Problem 5. Same settings as in Problem 3. Calculate the distance from $\phi(p_1)$ to the separation plane you find in the feature space.

Problem 6. Let P be a set of points in $\mathbb{R}^d$. Prove: the Gaussian kernel produces a kernel space where every point $p \in P$ is mapped to a point $\phi(p)$ satisfying $|\phi(p)| = 1$, namely, $\phi(p)$ is on the surface of an infinite-dimensional sphere.

Problem 7. For any a d-dimensional sphere centered at the origin of $\mathbb{R}^d$, we know that any set of $d + 1$ points on the sphere's surface can be shattered by the set of linear classifiers. Use this fact to prove that any finite set P of points in $\mathbb{R}^d$ can be linearly separated in the kernel space produced by the Gaussian kernel. (Hint: use the conclusion of Problem 6 and use the fact that the Gaussian kernel produces a kernel space of infinite dimensionality.)