Exercises for CSCI5010

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Problem 1*. Let $T : \mathbb{R}^d \to \mathbb{R}^d$ be an affine transformation. Given a set $P$ of points in $\mathbb{R}^d$, define by $T(P) = \{T(p) \mid p \in P\}$, namely, $T(P)$ is the set of points obtained by applying the affine transformation $T$ to $P$. Prove: if $R \subseteq P$ is an $\epsilon$-kernel of $P$, then $T(R)$ is an $\epsilon$-kernel of $T(P)$.

Hint: Given a directional vector $u$, the width of $P$ at direction $u$ can be calculated as

$$W_u(P) = \left( \max_{p \in P} u \cdot p \right) - \left( \min_{p \in P} u \cdot p \right)$$

where $u \cdot p$ is the dot product of vectors $u$ and $p$. To prove the claim, use your knowledge from linear algebra to figure out how a dot product would change under an affine transformation. Recall that an affine transformation is: $T(p) = Ap + b$ where $A$ is a $d \times d$ matrix, and both $p$ and $b$ are $d \times 1$ vectors.

Problem 2* ((1 − $\epsilon$)-Approximate Top-1 Search). Let $P$ be a set of points in $\mathbb{R}^d$ where $d$ is a constant, and each point has a positive coordinate on every dimension. We will view each point $p \in P$ as a $d$-dimensional vector $p = (p[1], p[2], \ldots, p[d])$ where $p[i]$ $(1 \leq i \leq d)$ is the $i$-th coordinate of $p$. Given a directional vector $u$ where $u[i] \geq 0$ for each $i \in [d]$, define

$$\text{top}_u(P) = \max_{p \in P} u \cdot p$$

where $u \cdot p$ is the dot product of vectors $u$ and $p$. Given $0 < \epsilon < 1$, describe an algorithm that computes in $O(n)$ expected time a subset $R \in P$ such that

- $|R| = O(1/\epsilon^d)$, and
- for any directional vector $u$, it holds that $\text{top}_u(R) \geq (1 - \epsilon) \cdot \text{top}_u(P)$.

Hint: Add the origin to $P$.

Problem 3. Prove the order-reversal property of dual transformation.

Problem 4. Prove the intersection preserving property of dual transformation.

Problem 5. Let $\ell_1$ and $\ell_2$ be two parallel non-vertical lines in the primal space $\mathbb{R}^2$. Prove: their vertical distance equals the distance of points $\ell_1^*$ and $\ell_2^*$ in the dual space.

Problem 6. Let $A$, $B$, $C$, and $D$ be four points in the primal space $\mathbb{R}^2$ that have distinct $x$ coordinates. Suppose that triangle $ABC$ has an area smaller than $ABD$. Let $\ell$ be the line passing points $A$ and $B$ in the primal space. Prove: in the dual space, point $\ell^*$ has a smaller vertical distance to line $C^*$ than to line $D^*$.

Note: The vertical distance from a point $(a, b)$ to a line $y = c_1 x - c_2$ equals $|b - (c_1 \cdot a - c_2)|$. 