Exercises for CSCI5010

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Problem 1. Let $P$ be a set of points in $\mathbb{R}^d$. Consider the following algorithm for finding an approximation of $\text{diam}(P)$ (i.e., the diameter of $P$).

algorithm
1. pick an arbitrary point $p \in P$
2. identify a point $q \in P$ maximizing $\text{dist}(p, q)$ (the Euclidean distance between $p$ and $q$)
3. return $\text{dist}(p, q)$

Prove: The algorithm returns a $\frac{1}{2}$-approximate answer, i.e., $\frac{1}{2} \cdot \text{diam}(P) \leq \text{dist}(p, q) \leq \text{diam}(P)$.

Problem 2. Let $P$ be a set of points in $\mathbb{R}^d$. Describe how to find in $O(n \log n + n/\epsilon^d)$ time a value $\Delta$ satisfying $\text{diam}(P) \leq \Delta \leq (1 + \epsilon) \cdot \text{diam}(P)$ for any $0 < \epsilon < 1$.

Problem 3*. Let $P$ be a set of 2D points. Let $p, q \in P$ be two points such that $\text{dist}(p, q) = \text{diam}(P)$. Prove: $p$ and $q$ must be vertices of the convex hull of $P$.

Hint: Use the fact that if a point $p$ is not a vertex of the convex hull, then every line $\ell$ passing $p$ has the property that there are points of $P$ falling on both sides of $\ell$.

Problem 4*. Let $G$ be a convex polygon. Define $p$ and $q$ as the two vertices of $G$ maximizing $\text{dist}(p, q)$. Prove: $G$ has an edge $e$ such that

- one of $p$ and $q$ is a vertex of $e$;
- if $p$ is a vertex of $e$, then $q$ is a vertex of $G$ having the maximum distance to the line $\ell$ passing $e$.

Hint: First find two parallel lines passing $p$ and $q$, respectively, and enclosing $G$ in between. Then, rotate these lines.

Remark: Problems 3 and 4 suggest that the (precise) diameter of a 2D point set $P$ can be found in $O(n \log n)$ time.

Problem 5 (reading exercise). Prove the correctness of the algorithm discussed in the lecture for computing a $t$-spanner graph for a set of points in $\mathbb{R}^d$.

Hint: If you do not want to think, read Pg 128-129 of the lecture notes. However, an inductive argument similar to the one for solving Problem 4 of Exercise List 6 suffices.
Problem 6 (reading exercise). Prove the correctness of the algorithm discussed in the lecture for computing a $(1 + \epsilon)$-approximate EMST for a set of points in $\mathbb{R}^d$.

Hint: If you do not want to think, read Pg 129-130 of the lecture notes.