Exercises for CSCI5010

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Let $P$ be a set of $n$ points in $\mathbb{R}^d$, where $d$ is a constant. Denote by $T$ the quadtree of $P$. In the lecture, we proved that our $s$-WSPD algorithm computes an $s$-WSPD of $P$ with $O(s^d \cdot n \cdot h)$ pairs, where $h$ is the height of $T$. Now, apply the same algorithm on the compressed quadtree tree $T_{\text{com}}$ of $P$. In this exercise, you will prove that the algorithm produces an $s$-WSPD of $O(s^d \cdot n)$ pairs.

We will apply the same charging strategy as introduced in the lecture. Every time the algorithm generates $\{u, v\}$ from $\{w, v\}$ by splitting $w$ (i.e., $w$ is the parent of $u$ in $T_{\text{com}}$), we charge the pair $\{u, v\}$ on $w$.

Solve the following problems.

**Problem 1.** For each node $z$ in $T_{\text{com}}$, we use $\text{level}(z)$ to denote the level of $z$ in the original quadtree $T$. Prove: $\text{level}(v) \geq \text{level}(w) \geq \text{level}(x)$.

Remark: Recall that if a node is at level $\ell$ of $T$, the node corresponds to a box with side length $1/2^\ell$ on each dimension. Essentially, you need to prove that the box of $v$ is no larger than that of $w$, which in turn is no larger than that of $x$.

Hint: Our algorithm always splits the “larger” node in a pair.

**Problem 2.** Fix a node $w$ in $T_{\text{com}}$ and a child $u$ of $w$. Prove: there are $O(s^d)$ nodes $v$ in $T_{\text{com}}$ satisfying (i) $\text{level}(w) = \text{level}(v)$ and (ii) $w$ is charged for the pair $\{u, v\}$.

**Problem 3.** Fix a node $w$ in $T_{\text{com}}$ and a child $u$ of $w$. Prove: there are $O(s^d)$ nodes $v$ in $T_{\text{com}}$ satisfying

- $\text{level}(w) = \text{level}(x)$ where $x$ is the parent of $v$ in $T_{\text{com}}$ and
- $w$ is charged for the pair $\{u, v\}$.

**Problem 4.** Fix a node $w$ in $T_{\text{com}}$ and a child $u$ of $w$. Let $S$ be the collection of nodes $v$ of $T_{\text{com}}$ satisfying

- $\text{level}(v) > \text{level}(w) > \text{level}(x)$ where $x$ is the parent of $v$ in $T_{\text{com}}$ and
- $w$ is charged for the pair $\{u, v\}$.

Consider any node $v \in S$ and let $x$ be the parent of $v$ in $T_{\text{com}}$. Identify the node in $T$ (the original quadtree) at level $\text{level}(w)$ on the path from $x$ to $v$ in $T$. We will refer to $\hat{v}$ the anchor node of $v$ with respect to $w$. Note that $\hat{v}$ has only a single child and does not exist in $T_{\text{com}}$ (i.e., $\hat{v}$ is removed by compression).
Prove: the nodes in $S$ have distinct anchor nodes with respect to $w$.

Hint: Which nodes on the path from $x$ to $v$ in $T$ have only one child?

**Problem 5.** Fix a node $w$ in $T_{com}$ and a child $u$ of $w$. Let $S$ be the collection of nodes $v$ of $T_{com}$ satisfying

- $\text{level}(v) > \text{level}(w) > \text{level}(x)$ where $x$ is the parent of $v$ in $T_{com}$ and
- $w$ is charged for the pair $\{u, v\}$.

Prove: $|S| = O(s^d)$.

Hint: Apply the Packing Lemma to bound the number of anchor nodes.

**Problem 6.** Prove: Each node $w$ of $T_{com}$ can be charged only $O(s^d)$ times.