Exercises

**Problem 1.** Consider the Voronoi diagram of a set \( P \) of points in \( \mathbb{R}^2 \). Prove: if a Voronoi vertex is incident to 4 Voronoi edges, then \( P \) has 4 points lying on the same circle.

**Problem 2.** \( P \) is a set of points in \( \mathbb{R}^2 \). Prove: if we take a point \( p \) from \( P \) uniformly at random, the number of Voronoi neighbors of \( p \) is \( O(1) \) in expectation.

**Problem 3.** Let \( P \) be a set of points in \( \mathbb{R}^2 \). Consider any point \( q \) in \( \mathbb{R}^2 \) (which may not be in \( P \)). Let \( p_1 \) be the nearest neighbor of \( q \) and \( p_2 \) be the second nearest neighbor (i.e., \( p_2 \) has the second smallest distance to \( q \) among all the points in \( P \)). Prove: \( p_2 \) must be a Voronoi neighbor of \( p_1 \).

(Hint: Argue there is a circle passing \( p_1, p_2 \) and containing no points of \( P \) in the interior.)

**Problem 4.** Prove the following for the triangulation of a point set \( P \) in \( \mathbb{R}^2 \):

- Every bounded face of the triangulation is a triangle.

  (Hint: If not, you can always add a diagonal. Recall our argument on polygon triangulation.)

- Every triangulation of \( P \) contains \( 2n - 2 - k \) triangles where \( n = |P| \) and \( k \) is the number of points on the convex hull boundary of \( P \).

  (Hint: Induction.)

**Problem 5.** Let \( ABC \) and \( DEF \) be two triangles. No triangle contains any vertex of the other. We know that segment \( AB \) intersects with segment \( DE \) in the interior. Prove: a segment in \( \{AC, BC\} \) must intersect a segment in \( \{DF, EF\} \).

![Diagram](image)

Remark: This completes our proof of the non-crossing lemma.