Let $P$ be a set of $n$ points in $\mathbb{R}^2$. We want to find a disc $D$ with the smallest radius to cover all the points in $P$. We refer to $D$ as the minimum enclosing disc (MED) of $P$ and denote it as $med(P)$. The lemma below explains why calling $D$ the MED is appropriate.

**Lemma 1.** There is only one disc with the smallest radius covering all the points in $P$.

**Proof.** Assume, on the contrary, that there are two such discs $D_1$ and $D_2$; see the figure below. Then, $P$ must be covered by the shaded area. Let $A$ and $B$ the intersection points of the two discs. Consider the disc $D$ centering at the midpoint $o$ of the segment $AB$ and having a radius equal to the length of segment $oA$. $D$ covers the shaded area (and hence, also $P$) but is smaller than $D_1$ and $D_2$, giving a contradiction.

Today we will learn a randomized algorithm for solving the problem in $O(n)$ expected time. As we will see, this is another beautiful application of backward analysis.

## 1 Geometric Facts

**Lemma 2.** The boundary of $med(P)$ passes at least two points of $P$.

**Proof.** Let $C$ be the boundary of $med(P)$. If $C$ passes no points of $P$, shrink $C$ infinitesimally to obtain a smaller disc still covering $P$, which contradicts the definition of $C$.

Suppose that $C$ passes only one point $p \in P$. Let $o$ be the center of $C$. Consider sliding a point $o'$ from $o$ towards $p$ infinitesimally, and look at the circle $C'$ centered at $o'$ with radius equal to the length of segment $o'p$. $C'$ is smaller than $C$ but still contains $P$ in the interior. This is also a contradiction.

The following geometric fact will be useful:
Lemma 3. Let $C_1$ and $C_2$ be two intersecting circles such that the radius of $C_2$ is larger than or equal to that of $C_1$. Let $\alpha$ be the area inside both circles. Let $p$ be an arbitrary point that is in $C_2$ but not in $C_1$. Then, there exists a circle $C$ that is smaller than $C_2$, larger than $C_1$, passes $p$, and contains the area $\alpha$.

The figure below gives an illustration of the lemma, where $C$ is the circle in dash line.

Proof. The lemma can be proved using basic geometry. We give only a sketch here (the complete proof is tedious and rudimentary).

Let us first discuss a relevant fact. Fix two distinct points $A, B$. Consider all the circles passing both $A$ and $B$. The centers of these circles must be on the perpendicular bisector of segment $AB$. Every such circle $C$ can be divided into (i) a left arc, which is the part of $C$ on the left of segment $AB$, and (ii) a right arc, which is the part of $C$ on the right. As the center $o$ of $C$ moves away from the midpoint $m$ of segment $AB$ towards right, the left arc “sweeps” towards segment $AB$, while the right arc “sweeps” away from the segment; furthermore, $C$ grows continuously. The behavior is symmetric when $o$ moves away from $m$ towards left.

Going back to the context of the lemma, let $A$ and $B$ be the intersection points of $C_1$ and $C_2$. Imagine morphing a circle $C$ from $C_2$ to $C_1$ while making sure that $C$ passes $A$ and $B$. Stop as soon as the right arc of $C$ hits $p$. This is the circle we are looking for.

2 Two Points Are Known

Let us first look at a variant of the MED problem. Let $p_1, p_2$ be two points in $P$ such that there is at least one disc which has $p_1, p_2$ on the boundary and covers the entire $P$. We want to find the smallest such disc, denoted as $med(P, \{p_1, p_2\})$. Algorithm 1 presents our solution in pseudocode. Its running time is clearly $O(n)$. To prove its correctness, it suffices to show:

Lemma 4. Define, for each $i \in [1, n]$, $P_i = \{p_1, ..., p_i\}$. For $i \geq 3$, let $D = med(P_{i-1}, \{p_1, p_2\})$. If $p_i$ is not covered by $D$, then the boundary of $med(P_i, \{p_1, p_2\})$ must pass $p_i$. 

2
**Algorithm 1: Two-Points-Fixed-MED**

```plaintext
/* suppose P = \{p_1, p_2, \ldots, p_n\} */
1 D ← the smallest disc covering p_1, p_2
2 for i = 3 to n do
3    if p_i not in D then
4        D ← the disc whose boundary passes p_1, p_2, p_i
5 return C
```

**Proof.** Let \( D' = med(P, \{p_1, p_2\}) \). Assume on the contrary that the boundary of \( D' \) does not pass \( p_i \). Hence, \( p_i \) falls inside \( D' \); see the figure below. The radius of \( D' \) cannot be smaller than that of \( D \) because the latter was the MED on \( P_{i-1} \) whereas \( D' \) is just one disc covering \( P_{i-1} \). The entire \( P_{i-1} \) must fall in the shaded area. By Lemma 3, there exists a disc smaller than \( D' \) covering \( P_{i-1} \), giving a contradiction.

![Diagram](image)

3 **One Point Is Known**

Next, we will generalize the two-points-fixed problem a bit. Let \( p_1 \) be a point in \( P \) such that there is at least one disc covering \( P \) whose boundary passes \( p_1 \). We want to find the smallest such circle, denoted as \( med(P, \{p_1\}) \).

**Algorithm 2: One-Point-Fixed-MED**

```plaintext
/* suppose P = \{p_1, p_2, \ldots, p_n\} */
1 randomly permute \( p_2, \ldots, p_n \)
2 D ← the smallest disc covering p_1, p_2
3 for i = 3 to n do
4    if p_i not in D then
5        D ← Two-Points-Fixed-MED(\{p_1, \ldots, p_i\}, \{p_1, p_i\})
6 return D
```

The algorithm’s correctness is ensured by:

**Lemma 5.** For \( i \geq 3 \), let \( D = med(P_{i-1}, \{p_1\}) \). If \( p_i \) is not covered by \( D \), then the boundary of \( med(P_i, \{p_1\}) \) must pass \( p_i \).

**Proof.** Left as an exercise.
Let us analyze the running time of the algorithm. Let \( t_i \) be the expected running time of the for-loop (Lines 3-5) for a specific \( i \). Thus, the total expected running time is \( O(\sum_{i=3}^{n} E[t_i]) \). Now, focus on \( t_i \) for a specific \( i \). Set \( D = med(P_i, \{p_1\}) \). We know that, besides \( p_1 \), the boundary of \( D \) is determined by at most 2 other points in \( P \) — let them be \( \pi_1, \pi_2 \) (if the boundary passes more than 2 points of \( P \) other than \( p_1 \), set \( \pi_1, \pi_2 \) to 2 arbitrary points of them). Hence, if \( p_i \neq \pi_1 \) and \( p_i \neq \pi_2 \), then \( t_i = O(1) \); otherwise, \( t_i = O(i) \) (Lemma 4). Standard backward analysis shows that \( E[t_i] \leq 2^{i-1}O(i) + O(1) = O(1) \). Therefore, the expected running time of Algorithm 2 is \( O(n) \), which subsumes the time of random permutation at Line 1.

4 No Point Is Known

We are ready to tackle the MED problem in its most general form:

```
Algorithm 3: MED(P)
/* suppose \( P = \{p_1, p_2, \ldots, p_n\} \) */
1 randomly permute \( p_1, \ldots, p_n \)
2 \( D \leftarrow \) the smallest disc covering \( p_1, p_2 \)
3 for \( i = 3 \) to \( n \) do
4     if \( p_i \) not in \( D \) then
5         \( D \leftarrow \) One-Point-Fixed-MED(\( \{p_1, \ldots, p_i\} \), \( \{p_i\} \))
6 return \( C \)
```

**Lemma 6.** For \( i \geq 3 \), let \( D = med(P_{i-1}) \). If \( p_i \) is not covered by \( D \), then the boundary of \( med(P_i) \) passes \( p_i \).

*Proof.* Left as an exercise.

We can once again apply backward analysis to prove that Algorithm 3 runs in \( O(n) \) expected time. The details are left as an exercise.