DFS and the Proof of White Path Theorem

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Let’s first go over the DFS algorithm through an example.

Suppose we start from the vertex $a$, namely $a$ is the root of DFS tree.
DFS

Firstly, set all the vertices to be white. Then, create a stack $S$, push the starting vertex $a$ into $S$ and color it gray. Create a DFS Tree with $a$ as the root. We also maintain the time interval $I(u)$ of each vertex $u$.

$S = (a)$.
Top of stack: \( a \), which has white out-neighbors \( b, c, f \). Suppose we access \( c \) first. Push \( c \) into \( S \).

\[
\begin{align*}
S &= (a, c).
\end{align*}
\]
After pushing $d$ into $S$:

$S = (a, c, d)$. 

DFS Tree  

Time Interval  

<table>
<thead>
<tr>
<th>Node</th>
<th>DFS Tree</th>
<th>Time Interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td></td>
<td>$I(a) = [1,$</td>
</tr>
<tr>
<td>$c$</td>
<td></td>
<td>$I(c) = [2,$</td>
</tr>
<tr>
<td>$d$</td>
<td></td>
<td>$I(d) = [3,$</td>
</tr>
</tbody>
</table>
Now $d$ tops the stack. It has white out-neighbors $e$, $f$ and $g$. Suppose we visit $g$ first. Push $g$ into $S$.

\[ S = (a, c, d, g). \]
After pushing $e$ into $S$:

$S = (a, c, d, g, e)$.

DFS and the Proof of White Path Theorem
e has no white out-neighbors. So pop it from $S$, and color it black. Similarly, $g$ has no white out-neighbors. Pop it from $S$, and color it black.

$S = (a, c, d)$. 

<table>
<thead>
<tr>
<th>DFS Tree</th>
<th>Time Interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>$I(a) = [1, \ ]$</td>
</tr>
<tr>
<td>$c$</td>
<td>$I(c) = [2, \ ]$</td>
</tr>
<tr>
<td>$d$</td>
<td>$I(d) = [3, \ ]$</td>
</tr>
<tr>
<td>$g$</td>
<td>$I(g) = [4, 7]$</td>
</tr>
<tr>
<td>$e$</td>
<td>$I(e) = [5, 6]$</td>
</tr>
</tbody>
</table>
Now $d$ tops the stack again. It still has a white out-neighbor $f$. So, push $f$ into $S$.

$$S = (a, c, d, f).$$
After popping $f$, $d$, $c$:

$$S = (a).$$
Now $a$ tops the stack again. It still has a white out-neighbor $b$. So, push $b$ into $S$.

$S = (a, b)$. 
After popping $b$ and $a$:

$$S = () .$$

Now, there is no white vertex remaining, our algorithm terminates.
Recall:

**White Path Theorem**: Let $u$ be a vertex in $G$. Consider the moment when $u$ is pushed into the stack in the DFS algorithm. Then, a vertex $v$ becomes a proper descendant of $u$ in the DFS-forest if and only if the following is true:

we can go from $u$ to $v$ by travelling only on white vertices.
Example

\[ S = \begin{array}{c}
\text{a} \\
\text{c}
\end{array} \]

DFS Tree

Final DFS Tree

DFS and the Proof of White Path Theorem
Lemma 1: Consider any two distinct vertices $u$ and $v$ in a DFS-tree. If $v$ is a descendant of $u$ in a DFS-tree, then $v$ enters the stack while $u$ is in the stack.

The proof is left to you.
Lemma 2: Consider any two distinct vertices $u$ and $v$ in a DFS-tree. If $v$ enters the stack while $u$ is in the stack, then $v$ is a descendant of $u$ in a DFS-tree.

The proof is left to you.
Proof of White Path Theorem

**White Path Theorem**: Let $u$ be a vertex in $G$. Consider the moment when $u$ is pushed into the stack in the DFS algorithm. Then, a vertex $v$ becomes a proper descendant of $u$ in the DFS-forest **if and only if** the following is true:

we can go from $u$ to $v$ by travelling only on white vertices.

**Proof**: The “only-if direction” ($\Rightarrow$): Let $v$ be a descendant of $u$ in the DFS tree. Let $\pi$ be the path from $u$ to $v$ in the tree. By Lemma 1, all the nodes on $\pi$ entered the stack after $u$. Hence, $\pi$ must be white at the moment when $u$ enters the stack.
The “if direction” ($\Leftarrow$): When $u$ enters the stack, there is a white path $\pi$ from $u$ to $v$. We will prove that all the vertices on $\pi$ must be descendants of $u$ in the DFS-forest.

Suppose that this is not true. Let $v'$ be the first vertex on $\pi$ — in the order from $u$ to $v$ — that is not a descendant of $u$ in the DFS-forest. Clearly $v' \neq u$. Let $u'$ be the vertex that precedes $v'$ on $\pi$; note that $u'$ is a descendant of $u$ in the DFS-forest.

By Lemma 2, $u'$ entered the stack after $u$. 
Consider the moment when \( u' \) turns black (i.e., \( u' \) leaving the stack). Node \( u \) must remain in the stack currently (first in last out).

1. The color of \( v' \) cannot be white.
   Otherwise, \( v' \) is a white out-neighbor of \( u \), which contradicts the fact that \( u' \) is turning black.

2. Hence, the color of \( v' \) must be gray or black.
   Recall that when \( u \) entered stack, \( v' \) was white. Therefore, \( v' \) must have been pushed into the stack while \( u \) was still in the stack. By the lemma on Slide 16, \( v' \) must be a descendant of \( u \). This, however, contradicts the definition of \( v' \).