Some Exercises on the “Three Basic Techniques”

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You have learned three basic techniques in algorithm design:

- Recursion
- Repeating (till success)
- Geometric Series.

In this tutorial, we will discuss some exercises that can be solved using these techniques.
Exercise 1

Recall that our RAM model has an atomic operation $\text{RANDOM}(x, y)$ which, given integers $x, y$, returns an integer chosen uniformly at random from $[x, y]$.

Suppose that you are allowed to call the operation only with $x = 1$ and $y = 128$. Describe an algorithm to obtain a uniformly random number between 1 and 100. Your algorithm must finish in $O(1)$ expected time.
Call RANDOM(1,128) and let $z$ be its return value. Output $z$ if it is in $[1, 100]$.

$z = 68$

Otherwise, repeat from the beginning.

$z = 120$
We need to call the operator at most twice in expectation because each time $z$ has probability $100/128$ to fall in the range we want. Therefore, our algorithm finishes in $O(1)$ expected time.
Exercise 2

Suppose that we enforce a harder constraint that you are allowed to call $\text{RANDOM}(x, y)$ only with $x = 0$ and $y = 1$. Describe an algorithm to generate a uniformly random number in $[1, n]$ for an arbitrary integer $n$. Your algorithm must finish in $O(\log n)$ expected time.
Suppose \( n \) is a power of 2; then how can we use recursion to solve this problem?

1. Set \( z = \text{RANDOM}(x, y) \).

2. If \( z = 0 \), we have a subproblem: generate a uniformly random number in the first half of the range;
   If \( z = 1 \), we have a subproblem: generate a uniformly random number in the second half of the range.

Considering the subproblem solved, we finish the algorithm.
Analysis of the Algorithm

\[ f(1) = O(1) \]
\[ f(n) \leq f(n/2) + O(1) , \text{ for } n > 1 \]

Thus, we have

\[ f(n) = O(\log n) \]

**Think:** Why does the algorithm require \( n \) to be a power of 2?
Next, we will extend our algorithm to support values of $n$ that are not powers of 2.

First, obtain the smallest power of 2 that is at least $n$.

- Try 1, 2, 4, ..., until reaching $m$ such that $n \leq m < 2n$. This takes $O(\log n)$ time.

We have known how to generate a uniformly random number $y$ in $[1, m]$ in $O(\log n)$ time.

If $y \leq n$, return $y$; otherwise, repeat the algorithm. At most 2 repeats are needed in expectation. The overall time is there $O(\log n)$ in expectation.
Exercise 3

Recall the $k$-selection problem:

You are given a set $S$ of $n$ integers in an array and an integer $k \in [1, n]$. Find the $k$-th smallest integer of $S$.

Suppose there is a deterministic algorithm $A_1$ which returns the median of $n$ integers in $O(n)$ time. Can you use $A_1$ as a blackbox to solve $k$-selection in $O(n)$ time?
Consider the following algorithm.

1. Get the median $v$ of $S$ from $A_1(S)$.

2. Divide $S$ into $S_1$ and $S_2$ where
   - $S_1 =$ the set of elements in $S$ less than or equal to $v$;
   - $S_2 =$ the set of elements in $S$ greater than $v$.

3. If $|S_1| \geq k$, then return $S' = S_1$ and $k' = k$; else return $S' = S_2$ and $k' = k - |S_1|$

Since $A_1$ is deterministic, we always succeed in obtaining a subproblem with size no larger than $\lceil \frac{|S|}{2} \rceil$. 
Analysis of the Algorithm

\[
f(1) = O(1) \\
f(n) \leq f(n/2) + O(n)
\]

Thus, \( f(n) = O(n) \).

What if \( A_1 \) returns the \( \lceil \frac{4}{5} n \rceil \)-th smallest integer of \( n \) integers in \( O(n) \) time. Can you still use \( A_1 \) as a blackbox to solve \( k \)-selection in \( O(n) \) time?
Instead of shrinking the size of subproblem by half, we shrink it by $\frac{4}{5}$.

We can still use $A_1$ to shrink the problem size by a constant factor. From the geometric series we know that the total cost will be $O(n)$.

**Think:** If $A_1$ returns the $\lceil \frac{99}{100} n \rceil$-th smallest integer of $n$ integers in $O(n)$ time, can you still use $A_1$ as a blackbox to solve $k$-selection in $O(n)$ time?
Exercise 4

Let’s still focus on the $k$-selection problem. In the lecture, we shrink the input size of the subproblem into at most $\frac{2}{3}n$. Now, we want to shrink the input size into at most $\frac{n}{2}$. Give an algorithm to achieve the purpose in $O(n)$ expected time.
A simple solution: run our $\frac{2n}{3}$-algorithm twice. The number of remaining elements becomes at most $\frac{4n}{9}$. 
Next, let us look at another way to achieve the purpose, assuming for simplicity that $n$ is a multiple of 4.

First, divide the rank space into 4 equal partitions.

![Diagram showing rank space divided into 4 equal partitions]

Second, take an element $p_1$ from $S$ uniformly at random. Repeat until $\text{rank}(p_1)$ is in range $\left[\frac{n}{4}, \frac{n}{2}\right]$.

![Diagram showing rank value for $p_1$]
Third, take an element $p_2$ from $S$ uniformly at random. Repeat until $\text{rank}(p_2)$ is in range $\left[\frac{1}{2}n, \frac{3}{4}n\right]$.

- If $k \leq \text{rank}(p_1)$, set $S' = \text{the set of elements in } S \text{ less than or equal to } p_1$, $k' = k$.
- If $\text{rank}(p_1) < k < \text{rank}(p_2)$, set $S' = \text{the set of elements in } S \text{ larger than } p_1 \text{ and smaller than } p_2$, $k' = k - \text{rank}(p_1)$.
- If $k \geq \text{rank}(p_2)$, set $S' = \text{the set of elements in } S \text{ larger than or equal to } p_2$, $k' = k - \text{rank}(p_2)$. 
In any case, we have $|S'| \leq \frac{n}{4} + \frac{n}{4} = \frac{n}{2}$.

In expectation, 4 repeats are needed for $p_1$, and 4 repeats for $p_2$ (think: why?).