CSCI3160: Quiz 2

Problem 1 (50%). Consider the optimal BST problem on \( S = \{1, 2, 3, 4\} \) and the weight array
\( W = (25, 15, 20, 50) \). Given integers \( a, b \in [1, 4] \), define
\[
\text{optavg}(a, b) = \begin{cases} 
0 & \text{if } a > b \\
\text{the smallest average cost of a BST on } \{a, a + 1, \ldots, b\} & \text{if } a \leq b
\end{cases}
\]

Some function values have been calculated for you:
\[
\begin{align*}
\text{optavg}(1, 1) &= 25 \\
\text{optavg}(1, 2) &= 55 \\
\text{optavg}(1, 3) &= 105 \\
\text{optavg}(2, 4) &= 135 \\
\text{optavg}(3, 4) &= 90 \\
\text{optavg}(4, 4) &= 50.
\end{align*}
\]
Prove: The optimal BST is not unique.

Solution. As derived in the lecture:
\[
\text{optavg}(1, 4) = \left( \sum_{i=1}^{4} W[i] \right) + \min_{r=1}^{4} \{ \text{optavg}(1, r - 1) + \text{optavg}(r + 1, b) \}.
\]

We enumerate all possibilities of the root’s key:

- If the root has key 1, the best BST has average cost \( 110 + \text{optavg}(2, 4) = 110 + 135 = 245 \).
- If the root has key 2, the best BST has average cost \( 110 + \text{optavg}(1, 1) + \text{optavg}(3, 4) = 110 + 25 + 90 = 225 \).
- If the root has key 3, the best BST has average cost \( 110 + \text{optavg}(1, 2) + \text{optavg}(4, 4) = 110 + 55 + 50 = 215 \).
- If the root has key 4, the best BST has average cost \( 110 + \text{optavg}(1, 3) = 110 + 105 = 215 \).

It thus follows that an optimal BST has average cost 215. As setting the root key to 3 or 4 can both yield an optimal BST, we know that there are at least two optimal BSTs.

Problem 2 (50%). Consider the directed graph \( G \) below.
Run the SCC (strongly connected components) algorithm taught in our lecture on this graph. Recall that the algorithm performs 3 steps:

- Step 1: Run DFS on $G$.
- Step 2: Create a new graph $G'$.
- Step 3: Run DFS on $G'$.

You must start the DFS of Step 1 from vertex 1. Answer the following questions:

(i) Indicate the vertices’ turn-black order obtained in Step 1.

(ii) Draw the graph $G'$.

(iii) Indicate all the SCCs output by Step 3, and the root of each DFS-tree produced in this step.

**Solution.** (i) The following are both correct turn-black orders:

- 6, 8, 5, 4, 3, 7, 2, 1
- 8, 6, 5, 4, 3, 7, 2, 1

(ii)

(iii) This solution holds for both turn-black orders given in (i).

First SCC: $\{1, 7, 2\}$, root vertex 1.
Second SCC: $\{3\}$, root vertex 3.
Third SCC: $\{4, 5, 6\}$, root vertex 4.
Fourth SCC: $\{8\}$, root vertex 8.