CSCI3160: Regular Exercise Set 9

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**Problem 1**. Prove the correctness of Dijkstra’s algorithm (when the edges have non-negative weights).

**Problem 2.** Consider again your proof for Problem 1. Point out the place that requires edge weights to be non-negative.

**Problem 3.** Consider a directed simple graph \( G = (V, E) \) where each edge \( e \in E \) has an arbitrary weight \( w(e) \) (which can be negative). It is known that \( G \) does not have negative cycles. Prove: given any vertices \( s, t \in V \), at least one shortest path from \( s \) to \( t \) is a simple path (i.e., no vertex appears twice on the path).

**Remark:** This implies that the path must have at most \( |V| - 1 \) edges.

**Problem 4***(SSSP in a DAG). Consider a simple acyclic directed graph \( G = (V, E) \) where each edge \( e \in E \) has an arbitrary weight \( w(e) \) (which can be negative). Solve the SSSP problem on \( G \) in \( O(|V| + |E|) \) time.

**Problem 5.** Let \( G = (V, E) \) be a simple directed graph where each edge \( e \in E \) carries a weight \( w(e) \), which can be negative. It is guaranteed that \( G \) has no negative cycles. Prove: given any vertices \( s, t \in V \), at least one shortest path from \( s \) to \( t \) is a simple path (i.e., no vertex appears twice on the path).

**Problem 6***(SSSP in a DAG). Let \( G = (V, E) \) be a simple directed graph where the weight of an edge \( (u, v) \) is \( w(u, v) \). Prove: the following algorithm correctly decides whether \( G \) has a negative cycle.

**algorithm** negative-cycle-detection
1. pick an arbitrary vertex \( s \in V \)
2. initialize \( dist(s) = 0 \) and \( dist(v) = \infty \) for every other vertex \( v \in V \)
3. **for** \( i = 1 \) **to** \( |V| - 1 \)
4.   **relax** all the edges in \( E \)
5. **for** each edge \( (u, v) \) \( \in E \)
6.   **if** \( dist(v) > dist(u) + w(u, v) \) **then**
7.     **return** “there is a negative cycle”
8. **return** “no negative cycles”