Detecting Negative Cycles with Floyd-Warshall

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We already know how to use the Floyd-Warshall (FW) algorithm to solve the APSP (all-pairs shortest path) problem when the input graph \( G = (V, E) \) contains no negative cycles.

Just like Bellman-Ford’s algorithm, the FW algorithm can also be used to **detect** negative cycles. For that purpose, FW runs in \( O(|V|^3) \) time, which is never better than the \( O(|V||E|) \) complexity of Bellman-Ford’s. Nevertheless, the correctness of FW is easier to understand.
Let us start by reviewing the Floyd-Warshall algorithm.

Define \( spdist(i, j \mid \leq k) \) as the smallest length of all paths from the vertex with id \( i \) to the vertex with id \( j \) that pass only intermediate vertices with ids \( \leq k \).

For \( k = 0 \), \( spdist(i, j \mid \leq 0) \) equals \( w(i, j) \) if \( E \) has an edge \((i, j)\), or \( \infty \), otherwise.

**Example:** Suppose \( a, b, ..., g \) have IDs 1, 2, ..., 7, respectively.

\[
spdist(3, 5 \mid 0) = -1, \quad spdist(3, 7 \mid 0) = \infty,
\]
Example

We use dynamic programming to compute $spdist(i, j \mid \leq k)$ for all $i, j, k$.

First, decide $spdist(i, j \mid \leq 0)$ for all $i, j \in [1, 7]$.
Example

\[ spdist(i, j \mid \leq k) = \]
\[ \min \left\{ spdist(i, j \mid \leq k - 1), \quad spdist(i, k \mid \leq k - 1) + spdist(k, j \mid \leq k - 1) \right\} \]

Then, compute \( spdist(i, j \mid \leq 1) \) for all \( i, j \in [1, 7] \). No changes.
Example

\[ spdist(i, j \mid \leq k) = \]
\[
\min \left\{ spdist(i, j \mid \leq k - 1),
\quad spdist(i, k \mid \leq k - 1) + spdist(k, j \mid \leq k - 1) \right\}
\]

Compute \( spdist(i, j \mid \leq 2) \) for all \( i, j \in [1, 7] \).
Example

\[ spdist(i, j \mid \leq k) = \]

\[ \min \left\{ \begin{array}{l}
spdist(i, j \mid \leq k - 1) \\
spdist(i, k \mid \leq k - 1) + spdist(k, j \mid \leq k - 1)
\end{array} \right\} \]

Compute \( spdist(i, j \mid \leq 3) \) for all \( i, j \in [1, 7] \).
**Example**

\[ \text{spdist}(i, j \mid \leq k) = \]

\[ \min \left\{ \text{spdist}(i, j \mid \leq k - 1), \text{spdist}(i, k \mid \leq k - 1) + \text{spdist}(k, j \mid \leq k - 1) \right\} \]

Compute \( \text{spdist}(i, j \mid \leq 4) \) for all \( i, j \in [1, 7] \).
Example

$$spdist(i, j \mid \leq k) =$$

$$\min \left\{ spdist(i, j \mid \leq k - 1), spdist(i, k \mid \leq k - 1) + spdist(k, j \mid \leq k - 1) \right\}$$

Compute $$spdist(i, j \mid \leq 5)$$ for all $$i, j \in [1, 7]$$.

<table>
<thead>
<tr>
<th>vertex $$v$$</th>
<th>$$a$$</th>
<th>$$b$$</th>
<th>$$c$$</th>
<th>$$d$$</th>
<th>$$e$$</th>
<th>$$f$$</th>
<th>$$g$$</th>
</tr>
</thead>
<tbody>
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</tr>
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<td>-1</td>
<td>0</td>
<td>$$\infty$$</td>
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<td>$$g$$</td>
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<td>$$\infty$$</td>
<td>$$\infty$$</td>
<td>2</td>
<td>$$\infty$$</td>
</tr>
</tbody>
</table>
Example

\[ \text{spdist}(i, j \mid \leq k) = \]
\[ \min \left\{ \begin{array}{l}
\text{spdist}(i, j \mid \leq k - 1) \\
\text{spdist}(i, k \mid \leq k - 1) + \text{spdist}(k, j \mid \leq k - 1)
\end{array} \right\} \]

Compute \( \text{spdist}(i, j \mid \leq 6) \) for all \( i, j \in [1, 7] \).
Example

\[ spdist(i, j \mid \leq k) = \]
\[ \min \left\{ \begin{array}{l}
spdist(i, j \mid \leq k - 1) \\
spdist(i, k \mid \leq k - 1) + spdist(k, j \mid \leq k - 1)
\end{array} \right. \]

Compute \( spdist(i, j \mid \leq 7) \) for all \( i, j \in [1, 7] \).

Now we are done.
Next, we will prove

\[ \text{spdist}(i, j \mid \leq k) = \min \left\{ \begin{array}{l}
\text{spdist}(i, j \mid \leq k - 1) \\
\text{spdist}(i, k \mid \leq k - 1) + \text{spdist}(k, j \mid \leq k - 1)
\end{array} \right\} \]

LHS \leq RHS is easy to prove. We will show only LHS \geq RHS.
Proof: The goal is to prove

\[ \text{spdist}(i, j \mid \leq k) \geq \]
\[ \min \left\{ \text{spdist}(i, j \mid \leq k - 1), \right. \]
\[ \left. \text{spdist}(i, k \mid \leq k - 1) + \text{spdist}(k, j \mid \leq k - 1) \right\} \]

Consider a path \( \pi \) from \( u \) to \( v \) that uses intermediate vertices only from \( \{1, 2, \cdots, k\} \) and has length \( \text{spdist}(u, v \mid \leq k) \).

If \( k \) is not an intermediate vertex of \( \pi \), then \( \pi \) has length at least \( \text{spdist}(u, v \mid \leq k - 1) \) (by definition).

Next, we discuss the case when \( k \) is an intermediate vertex of \( \pi \).
Goal: to prove

\[ spdist(i, j \mid \leq k) \geq \min \left\{ \begin{array}{ll}
spdist(i, j \mid \leq k - 1) \\
spdist(i, k \mid \leq k - 1) + spdist(k, j \mid \leq k - 1)
\end{array} \right. \]

Divide the case where \( k \) is an intermediate vertex of \( \pi \) into:

- **\( k \) appears only once on \( \pi \);**

  \[ spdist(i, k \mid \leq k - 1) \quad spdist(k, j \mid \leq k - 1) \]

- **\( k \) appears multiple times.**

  remove these cycles
What if the graph $G = (V, E)$ contains negative cycles?

A negative cycle: $a \rightarrow b \rightarrow c \rightarrow d \rightarrow a$.

Next, we will modify the FW algorithm for negative cycle detection.
Define a **simple path** from $u$ to $v$ as a path $\pi$ satisfying:

- $\pi$ starts from $u$ and ends at $v$.
- $u$ is not an intermediate vertex of $\pi$.
- $v$ is not an intermediate vertex of $\pi$.
- no intermediate vertex appears twice on $\pi$.

**Remark:** The simple-path definition allows $u = v$. 
We aim to find all-pairs shortest simple paths instead.

Re-define \( spdist(i, j \mid \leq k) \) as the smallest length of all simple paths from the vertex with id \( i \) to the vertex with id \( j \) that pass only intermediate vertices with ids \( \leq k \).

The following relationship still holds

\[
spdist(i, j \mid \leq k) = \min \left\{ \begin{array}{l}
spdist(i, j \mid \leq k - 1) \\
spdist(i, k \mid \leq k - 1) + spdist(k, j \mid \leq k - 1)
\end{array} \right\
\]

The proof is similar to the no-negative-cycle scenario and omitted.
Example:

We use dynamic programming to compute $spdist(i, j \mid \leq k)$ for all $i, j, k$.

For $k = 0$, $spdist(i, j \mid \leq 0)$ equals $w(i, j)$ if $E$ has an edge $(i, j)$, or $\infty$, otherwise.
Example

\[ spdist(i, j \mid \leq k) = \]
\[
\min \left\{ \begin{array}{l}
    spdist(i, j \mid \leq k - 1) \\
    spdist(i, k \mid \leq k - 1) + spdist(k, j \mid \leq k - 1)
\end{array} \right\}
\]

Then, compute \( spdist(i, j \mid \leq 1) \) for all \( i, j \in [1, 7] \).
Example

\[ \text{spdist}(i, j \mid \leq k) = \min \left\{ \text{spdist}(i, j \mid \leq k - 1), \text{spdist}(i, k \mid \leq k - 1) + \text{spdist}(k, j \mid \leq k - 1) \right\} \]

Then, compute \( \text{spdist}(i, j \mid \leq 2) \) for all \( i, j \in [1, 7] \).
Example

$$spdist(i, j \mid \leq k) = \min \begin{cases} 
spdist(i, j \mid \leq k - 1) \\
spdist(i, k \mid \leq k - 1) + spdist(k, j \mid \leq k - 1)
\end{cases}$$

Then, compute $spdist(i, j \mid \leq 3)$ for all $i, j \in [1, 7]$.
Example

\[ spdist(i, j \mid \leq k) = \]
\[ \min \begin{cases} 
  spdist(i, j \mid \leq k - 1) \\
  spdist(i, k \mid \leq k - 1) + spdist(k, j \mid \leq k - 1) 
\end{cases} \]

Then, compute \( spdist(i, j \mid \leq 4) \) for all \( i, j \in [1, 7] \).
Example

\[ spdist(i, j \mid \leq k) = \]
\[
\min \begin{cases} 
spdist(i, j \mid \leq k - 1) \\
spdist(i, k \mid \leq k - 1) + spdist(k, j \mid \leq k - 1)
\end{cases}
\]

Then, compute \( spdist(i, j \mid \leq 5) \) for all \( i, j \in [1, 7] \).
**Example**

\[
spdist(i, j | \leq k) = \min \left\{ \begin{array}{l}
spdist(i, j | \leq k - 1) \\
spdist(i, k | \leq k - 1) + spdist(k, j | \leq k - 1)
\end{array} \right.
\]

Then, compute \(spdist(i, j | \leq 6)\) for all \(i, j \in [1, 7]\).
**Example**

\[ \text{spdist}(i, j \mid \leq k) =  \]

\[
\min \left\{ \begin{array}{l}
\text{spdist}(i, j \mid \leq k - 1) \\
\text{spdist}(i, k \mid \leq k - 1) + \text{spdist}(k, j \mid \leq k - 1)
\end{array} \right. 
\]

Then, compute \( \text{spdist}(i, j \mid \leq 7) \) for all \( i, j \in [1, 7] \).
Check the diagonal results:

<table>
<thead>
<tr>
<th>vertex $v$</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$d$</th>
<th>$e$</th>
<th>$f$</th>
<th>$g$</th>
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<tbody>
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<td>$g$</td>
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<td>$4$</td>
<td>$3$</td>
<td>$2$</td>
<td>$5$</td>
<td></td>
</tr>
</tbody>
</table>

The graph $G$ has a negative cycle if and only if $spdist(u, u | \leq n) < 0$ for some $u \in V$. 
Next, we will prove that the algorithm is correct.
Proof of correctness. First, we prove that if $G$ has a negative cycle, then $spdist(u, u |\leq n) < 0$ for some $u \in V$.

Consider a negative cycle $C$. Let $u$ be the largest vertex on $C$, and $v$ be any other vertex on $C$.

Define $\pi_1$ as the path from $u$ to $v$ on $C$, and $\pi_2$ as the path from $v$ to $u$ on $C$. Length of $\pi_1$ is at least $spdist(u, v |\leq n - 1)$. Length of $\pi_2$ is at least $spdist(v, u |\leq n - 1)$. Thus,

$$spdist(u, u |\leq n) \leq spdist(u, v |\leq n - 1) + spdist(v, u |\leq n - 1) \leq \text{length of } C < 0.$$
Proof of correctness (cont.). Next, we prove that if \( spdist(u, u | \leq n) < 0 \) for some \( u \in V \), \( G \) has a negative cycle.

Let \( u \) be an arbitrary vertex satisfying \( spdist(u, u | \leq n) < 0 \). Then, there is a simple path from \( u \) to itself, with length \( spdist(u, u | \leq n) < 0 \). The simple path is a negative cycle.