Basic Techniques: Recursion, Repeating, and Geometric Series

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Today we will discuss three basic techniques of algorithm design:

- Recursion
- Repeating (till success)
- Geometric Series.
Recursion
Principle of recursion

When dealing with a subproblem (same problem but with a smaller input), consider it solved, and use the subproblem’s output to continue the algorithm design.
**Tower of Hanoi**

There are 3 rods A, B, and C.

On rod A, $n$ disks of different sizes are stacked in such a way that no disk of a larger size is above a disk of a smaller size.

The other two rods are empty.

![Diagram of the Tower of Hanoi](attachment:image.png)
Tower of Hanoi

**Permitted operation:** Move the top-most disk of a rod to another rod.

**Constraint:** No disk of a larger size can be above a disk of a smaller size.

**Goal:** Design an algorithm to move all the disks to rod B.
Subproblem: Same problem but with \( n - 1 \) disks.
Consider the subproblem solved (i.e., assume you already have an algorithm for it).

Now, solve the problem with \( n \) disks as follows:
Suppose that our algorithm performs $f(n)$ operations to solve a problem of size $n$. Clearly, $f(1) = 1$. By recursion, we can write

$$f(n) = 1 + 2 \cdot f(n - 1)$$

Solving this recurrence gives $f(n) = 2^n - 1$. 
Use recursion to “redesign” the following algorithms:

- Binary search
- Quick sort
Repeating till Success
The $k$-Selection Problem: You are given a set $S$ of $n$ integers in an array and an integer $k \in [1, n]$. Find the $k$-th smallest integer of $S$.

For example, suppose that $S = (53, 92, 85, 23, 35, 12, 68, 74)$ and $k = 3$. You should output 35.
The **rank** of an integer $v \in S$ is the number of elements in $S$ smaller than or equal to $v$.

For example, suppose that $S = (53, 92, 85, 23, 35, 12, 68, 74)$. Then, the rank of 53 is 4, and that of 12 is 1.

**Easy:** The rank of $v$ can be obtained in $O(|S|)$ time.
Consider the following task:

**Task:** Assume \( n \) to be a multiple of 3. Obtain a subproblem of size at most \( 2n/3 \) with exactly the same result as the original problem.

Our goal is to produce a set \( S' \) and an integer \( k' \) such that

- \( |S'| \leq 2n/3 \)
- \( k' \in [1, |S'|] \)
- The element with rank \( k' \) in \( S' \) is the element with rank \( k \) in \( S \).

We will give an algorithm to accomplish the task in \( O(n) \) expected time.
Consider the following algorithm.

1. Take an element $v \in S$ uniformly at random.
2. Divide $S$ into $S_1$ and $S_2$ where
   - $S_1 =$ the set of elements in $S$ less than or equal to $v$;
   - $S_2 =$ the set of elements in $S$ greater than $v$.
3. If $|S_1| \geq k$, then return $S' = S_1$ and $k' = k$;
   else return $S' = S_2$ and $k' = k - |S_1|$.

The algorithm succeeds if $|S'| \leq 2n/3$, or fails otherwise.

Repeat the algorithm until it succeeds.
**Lemma:** The algorithm succeeds with probability at least $1/3$.

**Proof:** The algorithm always succeeds when the rank of $v$ falls in $\left[\frac{n}{3}, \frac{2}{3}n\right]$ (think: why?). This happens with a probability at least $1/3$, by the fact that $v$ is taken from $S$ uniformly at random. □

In general, if an algorithm succeeds with a probability at least $c > 0$, then the number of repeats needed for the algorithm to succeed for the first time is at most $1/c$ in expectation.

We expect to repeat the algorithm at most 3 times before it succeeds. This implies that the expected running time is $O(n)$ (think: why?).
Geometric Series
A **geometric sequence** is an infinite sequence of the form

\[ n, cn, c^2 n, c^3 n, \ldots \]

where \( n \) is a positive number and \( c \) is a constant satisfying \( 0 < c < 1 \).

It holds in general that

\[
\sum_{i=0}^{\infty} c^i n = \frac{n}{1 - c} = O(n).
\]

The summation \( \sum_{i=0}^{\infty} c^i n \) is called a **geometric series**.

**Geometric series are extremely important for algorithm design.**
Consider again:

**The $k$-Selection Problem:** You are given a set $S$ of $n$ integers in an array and an integer $k \in [1, n]$. Find the $k$-th smallest integer of $S$. 
Using the repeating technique, now you should be able to convert the problem to a subproblem with size at most \( \lceil 2n/3 \rceil \) in \( O(n) \) expected time.

Now, apply the recursion technique. We have already obtained a (complete) algorithm solving the \( k \)-selection problem!

Think: How is this related to geometric series?
Let \( f(n) \) be the expected running time of our algorithm on an array of size \( n \).

We know:

\[
\begin{align*}
    f(1) & \leq O(1) \\
    f(n) & \leq O(n) + f(\lceil 2n/3 \rceil).
\end{align*}
\]

Solving the recurrence gives \( f(n) = O(n) \).