Three Exercises for Discussion

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Exercise 1 (Problem 1 of Regular Exercises List 1)

Let \( x \) be a real value. Define \( \lfloor x \rfloor \) to be the largest integer that does not exceed \( x \). For example, \( \lfloor 2.5 \rfloor = 2 \), whereas \( \lfloor 3 \rfloor = 3 \). Suppose that you are given an integer \( n \geq 2 \) in (a register of) the CPU. Write an algorithm to compute the value of \( \lfloor \log_2 n \rfloor \) in no more than \( 100 \log_2 n \) time.
Exercise 2

You are given a positive integer $n$ (that is stored in a register of the CPU). Design an algorithm to output the binary representation of $n$ in no more than $100 \lceil \log_2(n + 1) \rceil$ time. For example, the binary representations of 7 and 8 are 111 and 1000, respectively.
Solution to Exercise 2

Let $b_ib_{i-1}...b_0$ be the binary representation of $n$.

**Observation 1:** The integer division $\lfloor n/2 \rfloor$ gives $b_ib_{i-1}...b_2b_1$. Thus, the last bit $b_0$ can be calculated as $b_0 = n - \lfloor n/2 \rfloor \cdot 2$.

**Observation 2:** We can obtain $b_1$ by repeating the above on $b_ib_{i-1}...b_2b_1$.

Next, we analyze the running time. The binary form of $n$ has $\lceil \log_2(n + 1) \rceil$ digits. Therefore, we need to repeat for $\lceil \log_2(n + 1) \rceil$ times. We leave it to you to implement each repeat using no more than 100 atomic operations (this is trivial). Therefore, the total cost is at most $100\lceil \log_2(n + 1) \rceil$.
Exercise 3

You are given a positive integer $n$ (that is stored in a register of the CPU). Design an algorithm to calculate $\lfloor \sqrt{n} \rfloor$. Your algorithm should have a cost no more than $100\sqrt{n}$. 
Solution to Exercise 3

**Key observation:** \( \lfloor \sqrt{n} \rfloor \) is the largest integer \( x \geq 1 \) satisfying \( x^2 \leq n \).

**Algorithm:**

1. \( i \leftarrow 1 \)
2. \( \textbf{do} \)
3. \( \textbf{if} \ i^2 > n \textbf{ then return } i - 1 \)
4. \( i \leftarrow i + 1 \)