Binary Search Tree (Part 1)

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Today, we will introduce the **binary search tree** (BST). This lecture will focus on the **static** version of the BST (namely, without insertions and deletions), leaving the **dynamic** version to the next lecture.

Predecessor Search

Let S be a set of integers.

- A **predecessor query**: give an integer q, find its **predecessor** in S, which is the largest integer in S that does not exceed q;

Example: Suppose that $S = \{3, 10, 15, 20, 30, 40, 60, 73, 80\}$.

- The predecessor of 23 is 20
- The predecessor of 15 is 15
- The predecessor of 2 does not exist.

A binary search tree (BST) stores a set S of integers to support:

- the predecessor query;
- **Insertion**: adds a new integer to S;
- **Deletion**: removes an integer from S.

We will guarantee:

- $O(n)$ space consumption
- $O(\log n)$ time per predecessor query
- $O(\log n)$ time per insertion (next lecture)
- $O(\log n)$ time per deletion (next lecture)

where $n = |S|$.

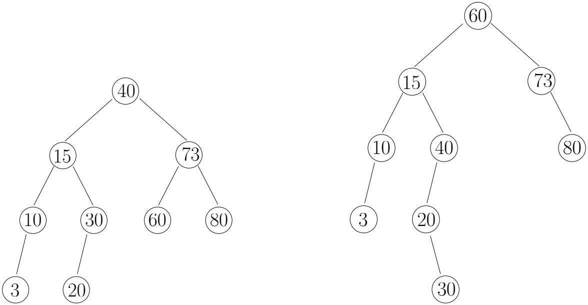
Binary Search Tree (BST)

We define a BST on a set S of n integers as a binary tree T satisfying all the following requirements:

- T has n nodes.
- Each node u in T stores a distinct integer in S, which is called the **key** of u.
- For every internal u:
  - its key is **larger than** all the keys in the **left** subtree;
  - its key is **smaller than** all the keys in the **right** subtree.

Example

Two possible BSTs on $S = \{3, 10, 15, 20, 30, 40, 60, 73, 80\}$.



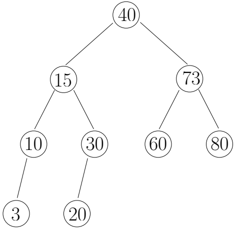
Balanced Binary Tree

A binary tree T is **balanced** if the following holds on every internal node u of T:

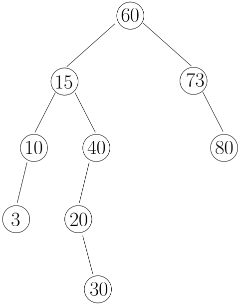
- The height of the left subtree of u differs from that of the right subtree of u by **at most 1**.

If u violates the above requirement, we say that u is **imbalanced**.

Example



Balanced



Imbalanced (nodes 40 and 60 are imbalanced)

Height of a Balanced Binary Tree

Theorem: A balanced binary tree with n nodes has height $O(\log n)$.

Proof: Denote the height as h. We will show that a balanced binary tree with height h must have $\Omega(2^{h/2})$ nodes.

This implies a constant $c > 0$ such that:

$$\begin{array}{rrcl} & n & \geq & c \cdot 2^{h/2} \\ \Rightarrow & 2^{h/2} & \leq & n/c \\ \Rightarrow & h/2 & \leq & \log_2(n/c) \\ \Rightarrow & h & = & O(\log n). \end{array}$$

Height of a Balanced Binary Tree

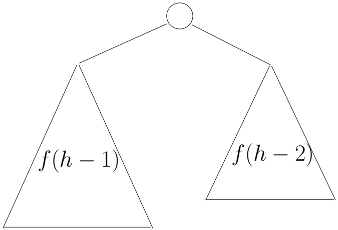
Let $f(h)$ be the minimum number of nodes in a balanced binary tree with height h. It is clear that:

$$\begin{aligned} f(1) &= 1 \\ f(2) &= 2 \end{aligned}$$

Height of a Balanced Binary Tree

In general, for $h \geq 3$:

$$f(h) \quad = \quad 1 + f(h-1) + f(h-2)$$



Height of a Balanced Binary Tree

When h is an even number:

$$
\begin{aligned}
f(h) &= 1 + f(h-1) + f(h-2) \\
&> 2 \cdot f(h-2) \\
&> 2^2 \cdot f(h-4) \\
&\dots \\
&> 2^{h/2-1} \cdot f(2) \\
&= 2^{h/2}
\end{aligned}
$$

Height of a Balanced Binary Tree

When h an odd number (i.e., $h \geq 3$):

$$\begin{aligned} f(h) &> f(h-1) \\ &> 2^{(h-1)/2} \\ &= 2^{h/2}/\sqrt{2} \\ &= \Omega(2^{h/2}) \end{aligned}$$

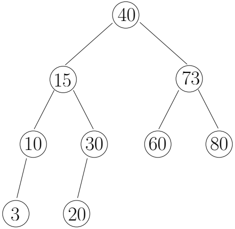
$\square$

Predecessor Query

Suppose that we have created a balanced BST T on a set S of n integers. A predecessor query with search value q can be answered by descending a single root-to-leaf path:

1. Set $p \leftarrow -\infty$ (p will contain the final answer at the end)
2. Set $u \leftarrow$ the root of T
3. If $u = \texttt{nil}$, then return p
4. If key of $u = q$, then set p to q, and return p
5. If key of $u > q$, then set u to the left child (now $u = \texttt{nil}$ if there is no left child), and repeat from Line 3.
6. Otherwise, set p to the key of u, set u to the right child, and repeat from Line 3.

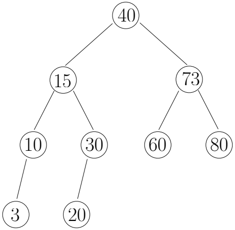
Example



Suppose that we want to find the predecessor of 35.

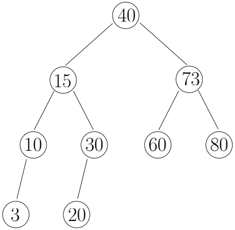
Start from u = root 40. Since $40 > 35$, the predecessor cannot be in the right subtree of 40. So we move to the left child of 40. Now u = node 15.

Example



Since $15 < 35$, the predecessor cannot be in the left subtree of 15. Update p to 15, because this is the predecessor of 35 so far, if we do not consider the right subtree of 15. Now, move u to the right child, namely, node 30.

Example



Since $30 < 35$, the predecessor cannot be in the left subtree of 30. Update p to 30. We need to move to the right child, but 30 does not have a right child. So the algorithm terminates here with $p = 30$ as the final answer.

Analysis of Predecessor Query Time

Obviously, we spend $O(1)$ time at each node visited. Since the BST is balanced, we know that its height is $O(\log n)$.

Therefore, the total query time is $O(\log n)$.

Successors

The opposite of predecessors are "successors".

Formally, the **successor** of an integer q in S is the smallest integer in S that is no smaller than q.

Suppose that $S = \{3, 10, 15, 20, 30, 40, 60, 73, 80\}$.

- The successor of 23 is 30
- The successor of 15 is 15
- The successor of 81 does not exist.

Finding a Successor

Given an integer q, a **successor query** returns the successor of q in S.

By symmetry, we know from the earlier discussion (on predecessor queries) that a predecessor query can be answered using a balanced BST in $O(\log n)$ time, where $n = |S|$.