Problem 1. Prove $30\sqrt{n} = O(\sqrt{n})$.

Solution. Set $c_1 = 30$ and $c_2 = 1$. The inequality $30\sqrt{n} \leq c_1\sqrt{n}$ holds for all $n \geq c_2$. This completes the proof.

Problem 2. Prove $\sqrt{n} = O(n)$.

Solution. Set $c_1 = 1$ and $c_2 = 1$. The inequality $\sqrt{n} \leq c_1n$ holds for all $n \geq c_2$. This completes the proof.

Problem 3. Let $f(n)$, $g(n)$, and $h(n)$ be functions of integer $n$. Prove: if $f(n) = O(g(n))$ and $g(n) = O(h(n))$, then $f(n) = O(h(n))$.

Solution. Since $f(n) = O(g(n))$, there exist constants $c_1, c_2$ such that for all $n \geq c_2$, it holds that

$$f(n) \leq c_1 g(n).$$

Similarly, since $g(n) = O(h(n))$, there exist constants $c'_1, c'_2$ such that for all $n \geq c'_2$, it holds that

$$g(n) \leq c'_1 h(n).$$

Set $c''_1 = c_1 c'_1$ and $c''_2 = \max\{c_2, c'_2\}$. From the above, we know that for all $n \geq c''_2$, it holds that

$$f(n) \leq c_1 g(n) \leq c_1 c'_1 h(n) = c''_1 h(n).$$

Therefore, $f(n) = O(h(n))$.

Problem 4. Prove $(2n + 2)^3 = O(n^3)$.

Solution. Set $c_1 = 4^3$ and $c_2 = 1$. The inequality $(2n + 2)^3 \leq c_1 n^3$ holds for all $n \geq c_2$. This completes the proof.

Problem 5. Prove or disprove: $4^n = O(2^n)$.

Solution. Consider the ratio $4^n/2^n$, which equals $2^n$. The ratio clearly goes to $\infty$ when $n$ tends to $\infty$. Therefore, the statement is incorrect.

Problem 6. Prove or disprove: $\frac{1}{n} = O(1)$.

Solution. Set $c_1 = 1$ and $c_2 = 1$. The inequality $1/n \leq c_1 \cdot 1$ holds for all $n \geq c_2$. This completes the proof.

Problem 7*. Prove that if $k \log k = \Theta(n)$, then $k = \Theta(n/\log n)$.

Solution. Since $k \log k = O(n)$, there exist constants $c_1, c_2$ such that $k \log k \leq c_1 n$ for all $n \geq c_2$. On the other hand, $k \log k = \Omega(n)$ indicates the existence of constants $c'_1, c'_2$ such that $k \log k \geq c'_1 n$ for all $n \geq c'_2$. Therefore, for all $n \geq \max\{c_2, c'_2\}$, we have:

$$c'_1 n \leq k \log k \leq c_1 n.$$  \hspace{1cm} (1)
Set \( c'_2 = \max\{c_1, c_2, c'_2\} \).

When \( n \geq \max\{c'_2, (1/c'_1)^2\} \), we derive from (1):

\[
\begin{align*}
\log_2(c'_1 n) &\leq \log_2(k \log_2 k) \leq \log_2(c_1 n) \\
\implies \log_2 c'_1 + \log_2 n &\leq \log_2 k + \log_2 \log_2 k \leq \log_2 c_1 + \log_2 n \\
\implies \left\{ \begin{array}{l}
\log_2 k \leq \log_2 c_1 + \log_2 n \leq 2 \log_2 n \\
2 \log_2 k \geq \log_2 k + \log_2 \log_2 k \geq \log_2 c'_1 + \log_2 n \geq \frac{1}{2} \log_2 n
\end{array} \right. \\
\implies \frac{\log_2 n}{4} &\leq \log_2 k \leq 2 \log_2 n.
\end{align*}
\]

(2)

Combining (1) and (2) leads to

\[
\begin{align*}
k \leq c_1 \frac{n}{\log_2 k} &\leq 4 c_1 \frac{n}{\log_2 n} \\
k \geq c'_1 \frac{n}{\log_2 k} &\geq \frac{c'_1}{2} \frac{n}{\log_2 n}
\end{align*}
\]

which means \( k = \Theta(n/\log n) \).

**Problem 8.** We can extend the big-O notation to multiple variables. In this problem, we will focus on two variables, but the idea extends to more variables in a straightforward manner.

Formally, let \( f(n, m) \) and \( g(n, m) \) be functions of variables \( n \) and \( m \) satisfying \( f(n, m) \geq 0 \) and \( g(n, m) \geq 0 \). We say \( f(n, m) = O(g(n, m)) \) if there exist constants \( c_1 \) and \( c_2 \) such that \( f(n, m) \leq c_1 \cdot g(n, m) \) holds for all \( n \geq c_2 \) and \( m \geq c_2 \).

Prove:

- \( n^2m + 100nm = O(n^2m) \).
- \( n^2m + 100nm^2 = O(n^2m + nm^2) \).

**Solution.** There exist constants \( c_1 = 101 \) and \( c_2 = 1 \) such that \( n^2m + 100nm \leq c_1 n^2m \) holds for all \( n \geq c_2 \) and \( m \geq c_2 \). Therefore, \( n^2m + 100nm = O(n^2m) \).

There exist constants \( c_1 = 100 \) and \( c_2 = 1 \) such that \( n^2m + 100nm \leq c_1 (n^2m + nm^2) \) holds for all \( n \geq c_2 \) and \( m \geq c_2 \). Therefore, \( n^2m + 100nm = O(n^2m + nm^2) \).