Connected Components and Dijkstra’s Algorithm

CSCI 2100 Teaching Team, Fall 2021
Outline

Today’s tutorial covers:

- find connected components using BFS and DFS.
- an example of Dijkstra’s Algorithm.
**Problem:** Let $G = (V, E)$ be an undirected graph. Our goal is to compute all the connected components (CC) of $G$.

A CC of $G$ includes a set $S \subseteq V$ of vertices such that:

- **(Connectivity)** any two vertices in $S$ are reachable from each other;
- **(Maximality)** not possible to add another vertex to $S$ while still satisfying the above requirement.

Output:

$\{a, b, c, d, f\}, \{g, e\}, \{h, i, j, k\}$
A Lemma on CCs

**Lemma:** Take an arbitrary vertex \( s \). The CC of \( s \) is the set \( S \) of vertices in \( G \) reachable from \( s \).

**Proof:**
- Connectivity: any two vertices in \( S \) can reach each other via \( s \).
- Maximality: any vertex outside \( S \) is unreachable from \( s \).
A BFS Solution

1. Run BFS on $G$ with a white source vertex
2. Output the vertex set of the BFS-tree
3. If there is still a white vertex in $G$, repeat from 1
Proof of Correctness

**Claim**: The vertex set $S$ of every BFS-tree is a CC of $G$.

**Proof**: Follows immediately because BFS finds all the vertices reachable from $s$. 

\[\square\]
A DFS Solution

1. Run DFS on $G$ with a white source vertex
2. Output the vertex set of the DFS-tree
3. If there is still a white vertex in $G$ repeat from 1
Proof of correctness

Claim: The vertex set $S$ of each DFS-tree is a CC of $G$.

Proof: Let $s$ be the source vertex of DFS. We will show that the DFS-tree contains all and only the vertices reachable from $s$.

Let $v$ be a vertex reachable from $s$. At the beginning of DFS, there is a white path from $s$ to $v$. By the white path theorem, $v$ must be in the subtree of $s$, namely, in the DFS-tree.

It is obvious that every vertex in the DFS-tree is reachable from $s$. 

\[ \square \]
Dijkstra’s Algorithm

The algorithm solves the single-source shortest-paths (SSSP) problem on a directed graph $G = (V, E)$ with positive edge weights.
Example

Suppose that the source vertex is $a$.

\[
\begin{array}{|c|c|c|}
\hline
\text{vertex } v & \text{dist}(v) & \text{parent}(v) \\
\hline
a & 0 & \text{nil} \\
b & \infty & \text{nil} \\
c & \infty & \text{nil} \\
d & \infty & \text{nil} \\
e & \infty & \text{nil} \\
f & \infty & \text{nil} \\
g & \infty & \text{nil} \\
h & \infty & \text{nil} \\
i & \infty & \text{nil} \\
\hline
\end{array}
\]

$F = \emptyset$ and

$P = \{a, b, c, d, e, f, g, h, i\}$.

Since $\text{dist}(a)$ is the smallest among those of vertices in $P$, pick $a$. 
**Example**

Relax the out-going edges of $a$:

$F = \{a\}$ (vertices finalized) and $P = \{b, c, d, e, f, g, h, i\}$.

Relaxing the edge $(a, b)$.

<table>
<thead>
<tr>
<th>vertex $v$</th>
<th>$dist(v)$</th>
<th>$parent(v)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>0</td>
<td>nil</td>
</tr>
<tr>
<td>$b$</td>
<td>$\infty \rightarrow 2$</td>
<td>nil $\rightarrow a$</td>
</tr>
<tr>
<td>$c$</td>
<td>$\infty$</td>
<td>nil</td>
</tr>
<tr>
<td>$d$</td>
<td>$\infty$</td>
<td>nil</td>
</tr>
<tr>
<td>$e$</td>
<td>$\infty$</td>
<td>nil</td>
</tr>
<tr>
<td>$f$</td>
<td>$\infty$</td>
<td>nil</td>
</tr>
<tr>
<td>$g$</td>
<td>$\infty$</td>
<td>nil</td>
</tr>
<tr>
<td>$h$</td>
<td>$\infty$</td>
<td>nil</td>
</tr>
<tr>
<td>$i$</td>
<td>$\infty$</td>
<td>nil</td>
</tr>
</tbody>
</table>


Example

Relax the out-going edges of $b$:

\[ F = \{a, b\} \text{ and} \]
\[ P = \{c, d, e, f, g, h, i\}. \]

Pick $b$ and relax $(b, d)$.
Example

Relax the out-going edges of \( d \):

\[
F = \{a, b, d\} \quad \text{and} \quad P = \{c, e, f, g, h, i\}.
\]

Pick \( d \) and relax \((d, c)\) and \((d, e)\).
Example

Relax the out-going edges of $e$: $F = \{a, b, d, e\}$ and $P = \{c, f, g, h, i\}$. 

$\begin{array}{|c|c|c|}
\hline
\text{vertex } v & \text{dist}(v) & \text{parent}(v) \\
\hline
a & 0 & \text{nil} \\
b & 2 & a \\
c & 12 & d \\
d & 5 & b \\
e & 6 & d \\
f & \infty \rightarrow 7 & \text{nil} \rightarrow e \\
g & \infty & \text{nil} \\
h & \infty & \text{nil} \\
i & \infty & \text{nil} \\
\hline
\end{array}$
Example

Relax the out-going edges of $f$:

$F = \{a, b, d, e, f\}$ and $P = \{c, g, h, i\}$.

<table>
<thead>
<tr>
<th>vertex $v$</th>
<th>$\text{dist}(v)$</th>
<th>$\text{parent}(v)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>0</td>
<td>nil</td>
</tr>
<tr>
<td>$b$</td>
<td>2</td>
<td>$a$</td>
</tr>
<tr>
<td>$c$</td>
<td>12 $\rightarrow$ 10</td>
<td>$d \rightarrow f$</td>
</tr>
<tr>
<td>$d$</td>
<td>5</td>
<td>$b$</td>
</tr>
<tr>
<td>$e$</td>
<td>6</td>
<td>$d$</td>
</tr>
<tr>
<td>$f$</td>
<td>7</td>
<td>$e$</td>
</tr>
<tr>
<td>$g$</td>
<td>$\infty$</td>
<td>nil</td>
</tr>
<tr>
<td>$h$</td>
<td>$\infty \rightarrow$ 11</td>
<td>nil $\rightarrow f$</td>
</tr>
<tr>
<td>$i$</td>
<td>$\infty$</td>
<td>nil</td>
</tr>
</tbody>
</table>
Example

Relax the out-going edges of $c$:

$$F = \{a, b, c, d, e, f\} \text{ and } P = \{g, h, i\}.$$
Example

Relax the out-going edges of $h$:

$F = \{ a, b, c, d, e, f, h \}$ and $P = \{ g, i \}$.
Example

Relax the out-going edges of $g$:

\[ F = \{ a, b, c, d, e, f, g, h \} \text{ and } P = \{ i \}. \]

<table>
<thead>
<tr>
<th>vertex $v$</th>
<th>$\text{dist}(v)$</th>
<th>$\text{parent}(v)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>0</td>
<td>nil</td>
</tr>
<tr>
<td>$b$</td>
<td>2</td>
<td>$a$</td>
</tr>
<tr>
<td>$c$</td>
<td>10</td>
<td>$f$</td>
</tr>
<tr>
<td>$d$</td>
<td>5</td>
<td>$b$</td>
</tr>
<tr>
<td>$e$</td>
<td>6</td>
<td>$d$</td>
</tr>
<tr>
<td>$f$</td>
<td>7</td>
<td>$e$</td>
</tr>
<tr>
<td>$g$</td>
<td>12</td>
<td>$h$</td>
</tr>
<tr>
<td>$h$</td>
<td>11</td>
<td>$f$</td>
</tr>
<tr>
<td>$i$</td>
<td>15 $\rightarrow$ 14</td>
<td>$h \rightarrow g$</td>
</tr>
</tbody>
</table>
Example

Relax the out-going edges of $i$:

$F = \{a, b, c, d, e, f, g, h, i\}$ and $P = \{}$.

Done.