Recursion
(Slides for ESTR2102)

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Recursion is an important technique in computer science for designing algorithms. Its principle is:

When dealing with a subproblem (same problem but with a smaller input), consider it solved.

We will discuss two examples in this lecture.
Tower of Hanoi

There are 3 rods: A, B, C.

On rod A, there are $n$ disks of different sizes, stacked in such a way that no disk of a larger size is above a disk of a smaller size.

The other two rods are empty.
Tower of Hanoi

**Permitted operation:** Move the top-most disk of a rod to another rod.

**Constraint:** No disk of a larger size can be above a disk of a smaller size.

![Diagram of Tower of Hanoi]

**Question:** How many operations are needed to move all disks to rod B?
Suppose that we have solved the problem with \( n - 1 \) disks. We can solve the problem with \( n \) disks as follows:
How many operations are needed by the algorithm?

Suppose that it is \( f(n) \). We have clearly \( f(1) = 1 \). Recursively:

\[
f(n) = 1 + 2 \cdot f(n - 1)
\]

Solving this recurrence gives: \( f(n) = 2^n - 1 \).
Different recursion strategies may result in very different running times. We will illustrate this in the next example.
Greatest Common Divisor (GCD)

Given two non-negative integers \( n \) and \( m \), find their GCD, denoted as \( GCD(n, m) \).

For example, \( GCD(24, 32) = 8 \). Note: \( GCD(0, 8) \) is also 8.

We want to design an algorithm in RAM with small running time.
Without loss of generality, assume $n \leq m$.

**Lemma:** If $n < m$, then $GCD(n, m) = GCD(n, m - n)$.

The proof is elementary and left to you.
GCD – Algorithm 1

Assume $n \leq m$.
If $n = m$, then return $n$.
Otherwise, return $GCD(n, m - n)$.

The running time can be as bad as $O(m)$. To see this, try computing $GCD(1, m)$.

Next, we will significantly improve the running time to $O(\log m)$. 
Greatest Common Divisor (GCD)

Without loss of generality, assume \( n \leq m \).

Define \( m \mod n = m - n \cdot \lfloor m/n \rfloor \).
Note that this is the remainder of \( m/n \).

Lemma: If \( n < m \), then \( \text{GCD}(n, m) = \text{GCD}(n, m \mod n) \).

The proof is elementary and left to you.
GCD – Algorithm 2 (Euclid’s Algorithm)

Assume $n \leq m$.
If $n = 0$, then return $m$
Otherwise, return $GCD(n, m \mod n)$.

Example

$GCD(24, 32) = GCD(24, 8) = GCD(0, 8) = 8$. 
Next, we will prove that the running time is $O(\log m)$.

Suppose we execute the “otherwise” line (see the previous slide) $h$ times. Let $n_i, m_i$ $(1 \leq i \leq m)$ be the two values of “$n$” and “$m$” at the $i$-th execution. Define $s_i = n_i + m_i$.

We will prove:

**Lemma:** For $i \geq 2$,
$$s_i \leq \frac{4}{5} \cdot s_{i-1}.$$ 

This implies $h = O(\log m)$ (**think: why?**). 

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GCD – Algorithm 2 (Euclid’s Algorithm)

**Lemma:** For $i \geq 2$, $s_i \leq \frac{4}{5} \cdot s_{i-1}$.

Essentially we need to prove: $n + m \mod n \leq \frac{4}{5}(n + m)$.

**Case 1:** $m \geq (3/2)n$.
Thus, $n + m \mod n < 2n = \frac{4}{5} \cdot \frac{5}{2}n \leq \frac{4}{5}(n + m)$.

**Case 2:** $m < (3/2)n$.
Thus, $n + m \mod n < n + n/2 = \frac{3}{2}n = \frac{3}{4} \cdot 2n \leq \frac{3}{4}(n + m)$.

We now conclude the proof.
Lowest Common Multiplier (GCM)

Given two non-negative integers \( n \) and \( m \), find their LCM.

For example, the LCM of 24 and 32 is 96.

**Think:** How to solve the problem in \( O(\log n) \) time using the GCD algorithm?