Example and proof of Dijkstra's Algorithm

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Adapted from the slides of the previous offerings of the course.

Dijkstra's Algorithm

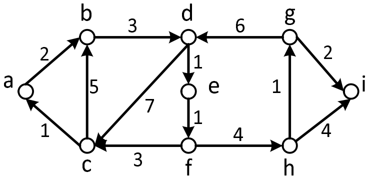
The algorithm solves the single-source shortest-paths (SSSP) problem on a directed graph $G = (V, E)$ with positive edge weights.

Let $V' \subseteq V$ be the current set of vertices whose shortest paths from the source vertex s have been found and $S = V \backslash V'$.

The crucial part of the algorithm is the edge relaxation idea. Essentially, we will prove this later, it is to maintain, for each $v \in S$, the "current shortest" distance from s only through the vertices in V'.

Example

Suppose that the source vertex is a.

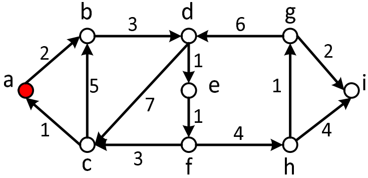


vertex v	$dist(v)$	$parent(v)$
a	0	nil
b	∞	nil
c	∞	nil
d	∞	nil
e	∞	nil
f	∞	nil
g	∞	nil
h	∞	nil
i	∞	nil

$V' = \emptyset$ and
$S = \{a, b, c, d, e, f, g, h, i\}$.
Since $dist(a)$ is the smallest among those of vertices in S, pick a.

Example

Relax the out-going edges of a:

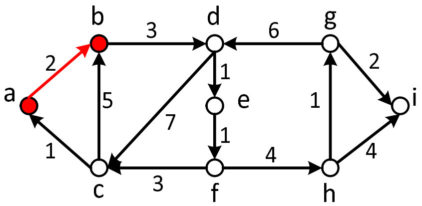


vertex v	$dist(v)$	$parent(v)$
a	0	nil
b	$\infty \to 2$	nil $\to$ a
c	∞	nil
d	∞	nil
e	∞	nil
f	∞	nil
g	∞	nil
h	∞	nil
i	∞	nil

$V' = \{a\}$ and
$S = \{b, c, d, e, f, g, h, i\}$.
The "current shortest" distance of b from a only through the vertices in V' is updated. After then, $dist(b)$ is the smallest among those of vertices in S. Pick b.

Example

Relax the out-going edges of b:

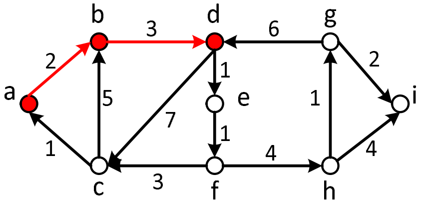


vertex v	$dist(v)$	$parent(v)$
a	0	nil
b	2	a
c	∞	nil
d	$\infty \rightarrow 5$	nil $\rightarrow$ b
e	∞	nil
f	∞	nil
g	∞	nil
h	∞	nil
i	∞	nil

$V' = \{a, b\}$ and
$S = \{c, d, e, f, g, h, i\}$.
Similarly, update the "current shortest" distance of d from a only through the vertices in V'. And $dist(d)$ is the smallest among those in S. Pick d.

Example

Relax the out-going edges of d:

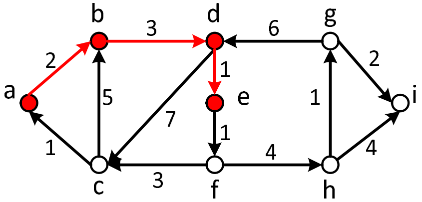


vertex v	$dist(v)$	$parent(v)$
a	0	nil
b	2	a
c	$\infty \to 12$	nil $\to d$
d	5	b
e	$\infty \to 6$	nil $\to d$
f	∞	nil
g	∞	nil
h	∞	nil
i	∞	nil

$V' = \{a, b, d\}$ and
$S = \{c, e, f, g, h, i\}$.
Since after the updates, $dist(e)$ is the smallest among those in S, pick e.

Example

Relax the out-going edges of e:

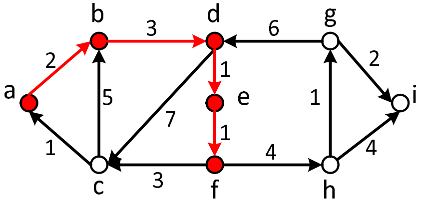


vertex v	$dist(v)$	$parent(v)$
a	0	nil
b	2	a
c	12	d
d	5	b
e	6	d
f	$\infty \to 7$	nil $\to e$
g	∞	nil
h	∞	nil
i	∞	nil

$V' = \{a, b, d, e\}$ and
$S = \{c, f, g, h, i\}$.

Example

Relax the out-going edges of f:

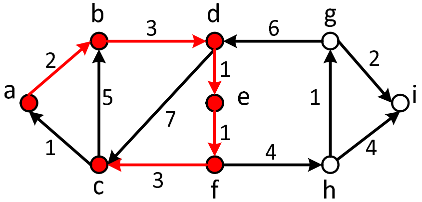


vertex v	$dist(v)$	$parent(v)$
a	0	nil
b	2	a
c	$12 \rightarrow 10$	$d \rightarrow f$
d	5	b
e	6	d
f	7	e
g	∞	nil
h	$\infty \rightarrow 11$	nil $\rightarrow f$
i	∞	nil

$V' = \{a, b, d, e, f\}$ and
$S = \{c, g, h, i\}$.

Example

Relax the out-going edges of c:

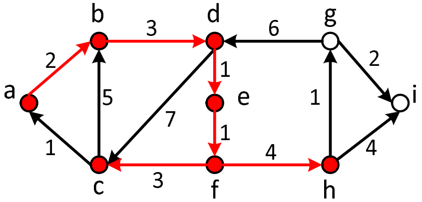


vertex v	$dist(v)$	$parent(v)$
a	0	nil
b	2	a
c	10	f
d	5	b
e	6	d
f	7	e
g	∞	nil
h	11	f
i	∞	nil

$V' = \{a, b, c, d, e, f\}$ and
$S = \{g, h, i\}$.

Example

Relax the out-going edges of h:

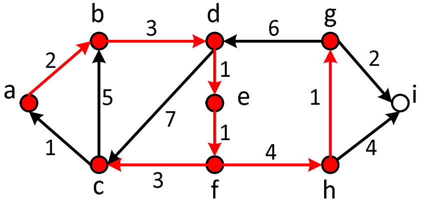


vertex v	$dist(v)$	$parent(v)$
a	0	nil
b	2	a
c	10	f
d	5	b
e	6	d
f	7	e
g	$\infty \to 12$	nil $\to h$
h	11	f
i	$\infty \to 15$	nil $\to h$

$V' = \{a, b, c, d, e, f, h\}$ and
$S = \{g, i\}$.

Example

Relax the out-going edges of g:

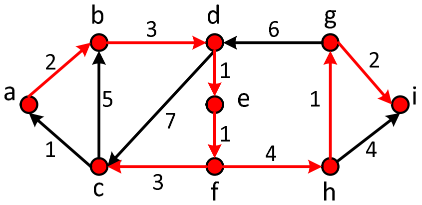


vertex v	$dist(v)$	$parent(v)$
a	0	nil
b	2	a
c	10	f
d	5	b
e	6	d
f	7	e
g	12	h
h	11	f
i	$15 \to 14$	$h \to g$

$V' = \{a, b, c, d, e, f, g, h\}$ and $S = \{i\}$.

Example

Relax the out-going edges of i:



vertex v	$dist(v)$	$parent(v)$
a	0	nil
b	2	a
c	10	f
d	5	b
e	6	d
f	7	e
g	12	h
h	11	f
i	14	g

$V' = \{a, b, c, d, e, f, g, h, i\}$ and
$S = \{\}$.
Done.

Correctness of Dijkstra's Algorithm

Lemma: When vertex v is removed from S, $dist(v)$ equals precisely the shortest path distance—denoted as $spdist(v)$—from s to v.

The correctness of Dijkstra's algorithm follows from the lemma.

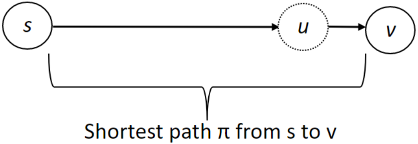
Correctness of Dijkstra's Algorithm

We will prove the claim by induction on the sequence of vertices removed.

- Base case:
  This is obviously true for the first vertex removed, which is s itself with $dist(s) = 0$.
- Inductive:
  Assume the claim is true with respect to all the vertices already removed. Let v be the next node to be removed. Need to prove $dist(v) = spdist(v)$.

Correctness of Dijkstra's Algorithm

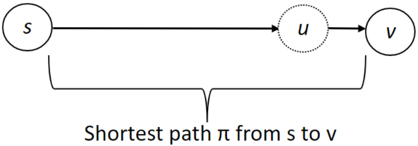
Consider an arbitrary shortest path π from s to v. Let u be the vertex right before v on π.



Claim: u must have been removed from S.

Our target lemma follows from the above claim because, by our inductive assumption, $dist(u) = spdist(u)$ when u was removed. Then, the algorithm relaxed the edge (u, v), which must have set $dist(v) = spdist(u) + w(u, v) = spdist(v)$.

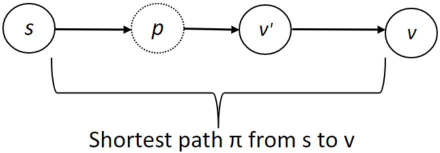
Correctness of Dijkstra's Algorithm



Stronger claim: All the nodes on π from s to u must have been removed.

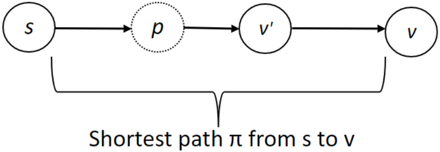
Correctness of Dijkstra's Algorithm

We will prove the stronger claim by contradiction.



Suppose the statement is not true. When v is to be removed from S, another vertex on π — let it be v' — still remains in S. Define p as the vertex right before v' on π.

Correctness of Dijkstra's Algorithm



By the inductive assumption, $dist(p) = spdist(p)$ when p was removed. Hence, after relaxing the edge (p, v'), we have $dist(v') = spdist(p) + w(p, v') = spdist(v')$.

But this means $dist(v') = spdist(v') < spdist(v) \leq dist(v)$!

Hence, v' should be the next vertex to be removed from S, contradicting the definition of v.