Breadth First Search

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In this lecture, we will discuss a simple algorithm—called **breadth first search**—to traverse all the nodes and edges in a graph once. Our discussion will focus on directed graphs, because the extension to undirected graphs is straightforward.

To make the discussion more interesting, we will cast it in a concrete problem: **single source shortest path** (SSSP) with unit weights.

Shortest Path

Let $G = (V, E)$ be a directed graph.

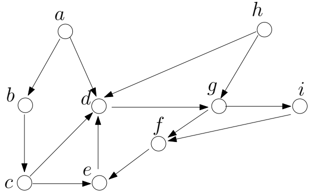
A **path** in G is a sequence of edges $(v_1, v_2), (v_2, v_3), ..., (v_\ell, v_{\ell+1})$, for some integer $\ell \geq 1$, which is called the **length** of the path. The path is said to be **from** v_1 **to** $v_{\ell+1}$.

- Sometimes, we will also denote the path as $v_1 \to v_2 \to ... \to v_{\ell+1}$.

Given two vertices $u, v \in V$, a **shortest path** from u to v is a path of the minimum length from u to v.

If there is no path from u to v, then v is **unreachable** from u.

Example



There are several paths from a to g:

- $a \to b \to c \to d \to g$ (length 4)
- $a \to b \to c \to e \to d \to g$ (length 5)
- $a \to d \to g$ (length 2)

The last one is a shortest path. In this case, the shortest path is unique. Note that h is unreachable from a.

Single Source Shortest Path (SSSP) with Unit Weights

Let $G = (V, E)$ be a directed graph, and s be a vertex in V. The goal of the **SSSP problem** is to find, for every other vertex $t \in V \setminus \{s\}$, a shortest path from s to t, unless t is unreachable from s.

Next, we will describe the breadth first search (BFS) algorithm to solve the problem in $O(|V| + |E|)$ time, which is clearly optimal (because any algorithm must at least see every vertex and every edge once in the worst case).

At first glance, this may look surprising because the total length of all the shortest paths may reach $\Omega(|V|^2)$, even when $|E| = O(|V|)$ (can you give such an example?)! So shouldn't the algorithm need $\Omega(|V|^2)$ time just to output all the shortest paths in the worst case?

The answer, interestingly, is no. As will see, BFS encodes all the shortest paths in a BFS tree compactly, which uses only $O(|V|)$ space, and can be output in $O(|V| + |E|)$ time.

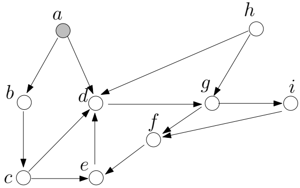
BFS

At the beginning, color all vertices in the graph **white** and create an empty BFS tree T.

Create a queue Q. Insert the source vertex s into Q and color it **gray** (which means "in the queue"). Make s the root of T.

Example

Suppose that the source vertex is a.



BFS tree

a

$Q = (a)$.

BFS

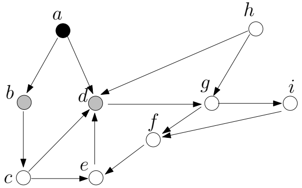
Repeat the following until Q is empty.

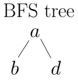
1. De-queue from Q the first vertex v.
2. For every out-neighbor u of v that is still white:
   - 2.1 En-queue u into Q, and color u **gray**.
   - 2.2 Make u a child of v in the BFS tree T.
3. Color v **black** (meaning that v is done).

> BFS behaves like "spreading a virus", as we will see from our running example.

Running Example

After de-queueing a:

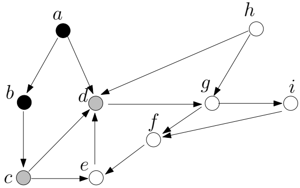


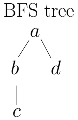


$Q = (b, d)$.

Running Example

After de-queueing b:

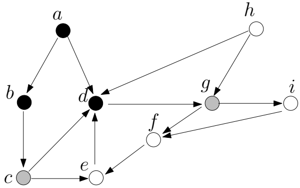


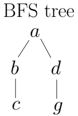


$Q = (d, c)$.

Running Example

After de-queueing d:

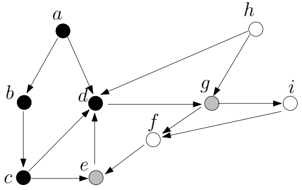


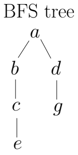


$Q = (c, g)$.

Running Example

After de-queueing c:



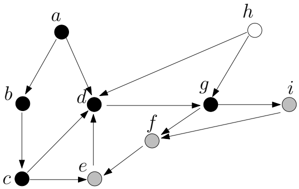


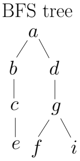
$Q = (g, e)$.

Note: d is not en-queued again because it is black.

Running Example

After de-queueing g:

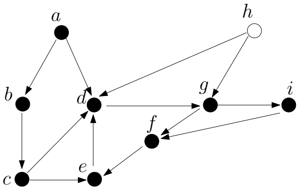


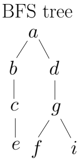


$Q = (e, f, i)$.

Running Example

After de-queueing e, f, i:



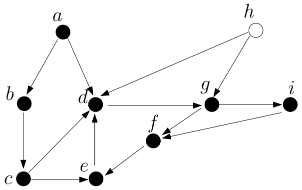


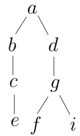
$Q = ()$.

This is the end of BFS. Note that h remains white—we can conclude that it is not reachable from a.

Running Example

Where are the shortest paths?





The shortest path from a to any vertex, say, x is simply the path from a to node x in the BFS tree!

- The proof will be left as an exercise.

Time Analysis

When a vertex v is de-queued, we spend $O(1 + d^+(v))$ time processing it, where $d^+(v)$ is the out-degree of v.

Clearly, every vertex enters the queue at most once.

The total running time of BFS is therefore

$$O\left(\sum_{v \in V} (1 + d^+(v))\right) = O(|V| + |E|).$$