More on Binary Heaps

CSCI2100 Tutorial 9

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Adapted from the slides of the previous offerings of the course

Introduction

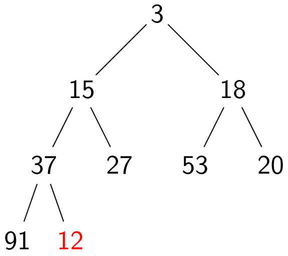
In the previous lectures, we have implemented the priority queue (which supports `insert(e)` and `delete-min` operations) using a data structure called the binary heap and achieved the following guarantees:

- $O(n)$ space consumption
- $O(\log n)$ insertion time
- $O(\log n)$ delete-min time

In this tutorial, we will try to enhance our understanding of the binary heap through some examples and exercises.

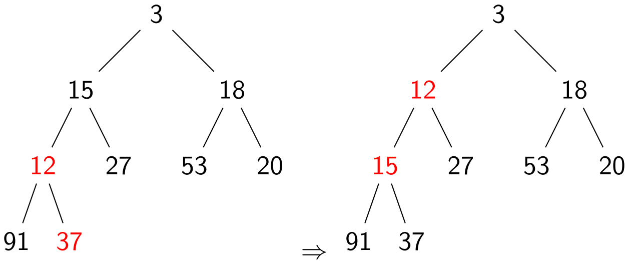
Example on Insertion

Assume that we want to insert 12 into the following binary heap. Fisrt, add 12 as a leaf, making sure that we still have a complete binary tree.



Example on Insertion

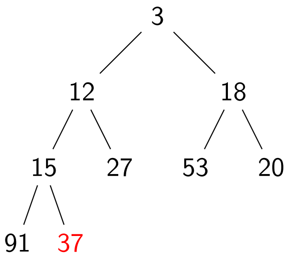
Then we fix the violations caused by this newly added element.



No more violations, insertion complete. An insertion can be processed in $O(\log n)$ time.

Example on Delete-min

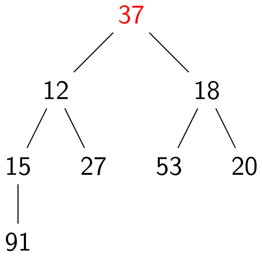
Assume that we want to perform delete-min from this binary heap below:



First, find the rightmost leaf at the bottom level, namely, 37.

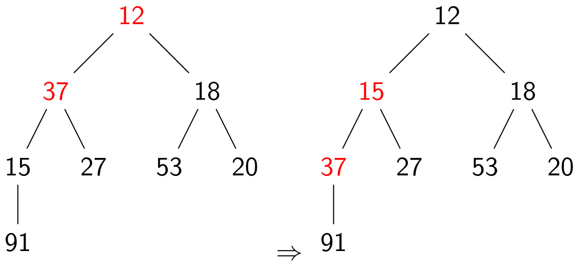
Example on Delete-min

Remove this leaf, but place the value 37 in the root.



Example on Delete-min

Then we fix the violations caused by 37.



No more violations, delete-min complete. A delete-min can be processed in $O(\log n)$ time.

Regular Exercise 8 Problem 4

Problem

Suppose that we have k sorted arrays (in ascending order) $A_1, A_2, \cdots, A_k$ of integers. Let n be the total number of integers in those arrays. Describe an algorithm to produce an array that sorts all the n integers in ascending order in $O(n \log k)$ time.

Example

Suppose that $k = 3$, and the 3 arrays are as follows:

A_1:

2	23	32	35	37

A_2:

5	10

A_3:

33	58	82

Then you should produce an array B as below in $O(n \log k)$ time.

B:

2	5	10	23	32	33	35	37	58	82

Regular Exercise 8 Problem 4

Solution

Insert the smallest elements of each array into a binary heap H. This takes $O(k \log k)$ time. Then, repeat the following until H is empty:

- Perform a `delete-min`. Let e be the element fetched.
- Append e to the output array.
- If e comes from A_i (for some i), obtain the next element from A_i, and insert it into H. If A_i has been exhausted, then do nothing.

Regular Exercise 8 Problem 4

Example

Suppose that $k = 3$, and the 3 arrays are as follows:

A_1:

2	23	32	35	37

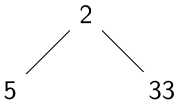
A_2:

5	10

A_3:

33	58	82

First, we insert the smallest elements of each array into a binary heap H:



Initially, the output array B is empty.

B:

Regular Exercise 8 Problem 4

Example

A_1:

2	23	32	35	37

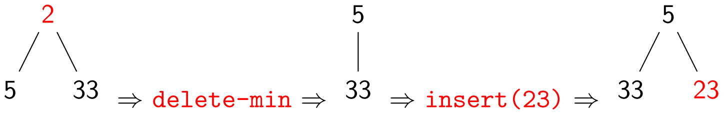
A_2:

5	10

A_3:

33	58	82

Then, we perform a `delete-min` on H, and fetch $e = 2$, then append 2 to the output array B. Since e comes from A_1, we obtain the next element 23 from A_1, and insert it into H.



Output array B:

2									

Regular Exercise 8 Problem 4

Example

A_1:

2	23	32	35	37

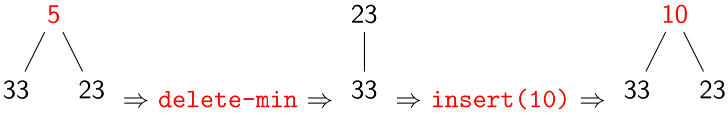
A_2:

5	10

A_3:

33	58	82

Perform another `delete-min` on the new H, and fetch $e = 5$, then append 5 to the output array B. Since e comes from A_2, we obtain the next element 10 from A_2, and insert it into H.



Output array B:

2	5								

A_1:

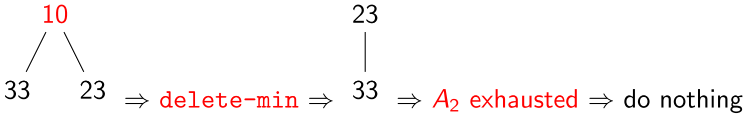
2	23	32	35	37

A_2:

5	10

A_3:

33	58	82



Output array B:

2	5	10							

A_1:

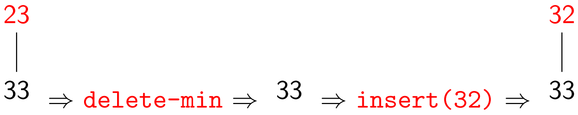
2	23	32	35	37

A_2:

5	10

A_3:

33	58	82



Output array B:

2	5	10	23						

A_1: | 2 | 23 | 32 | 35 | 37 | A_2: | 5 | 10 | A_3: | 33 | 58 | 82 |

32
|
33 $\Rightarrow$ `delete-min` $\Rightarrow$ 33 $\Rightarrow$ `insert(35)` $\Rightarrow$ 33
|
35

Output array B: | 2 | 5 | 10 | 23 | 32 | | | | | |

A_1: | 2 | 23 | 32 | 35 | 37 | A_2: | 5 | 10 | A_3: | 33 | 58 | 82 |

33
|
35 $\Rightarrow$ `delete-min` $\Rightarrow$ 35 $\Rightarrow$ `insert(58)` $\Rightarrow$ 35
|
58

Output array B: | 2 | 5 | 10 | 23 | 32 | 33 | | | | |

A_1:

2	23	32	35	37

A_2:

5	10

A_3:

33	58	82

35

58 $\Rightarrow$ `delete-min` $\Rightarrow$ 58 $\Rightarrow$ `insert(37)` $\Rightarrow$

37

58

Output array B:

2	5	10	23	32	33	35			

A_1:

2	23	32	35	37

A_2:

5	10

A_3:

33	58	82

37

58 $\Rightarrow$ `delete-min` $\Rightarrow$ 58 $\Rightarrow$ A_1 exhausted $\Rightarrow$ do nothing

Output array B:

2	5	10	23	32	33	35	37		

A_1: | 2 | 23 | 32 | 35 | 37 | A_2: | 5 | 10 | A_3: | 33 | 58 | 82 |

58 $\Rightarrow$ `delete-min` $\Rightarrow$ `insert(82)` $\Rightarrow$ 82

Output array B:

2	5	10	23	32	33	35	37	58	

A_1: | 2 | 23 | 32 | 35 | 37 | A_2: | 5 | 10 | A_3: | 33 | 58 | 82 |

82 $\Rightarrow$ `delete-min` $\Rightarrow$ no more insertions $\Rightarrow$ H is empty $\Rightarrow$ stop

Finally, we produce the output array B with all the $n = 10$ elements sorted in ascending order:

Output array B:

2	5	10	23	32	33	35	37	58	82

Cost Analysis

- Insert the smallest elements of each array into a binary Heap H takes $O(k \log k)$ time.
- Each delete-min and insertion require $O(\log k)$ time.
  - Since H has at most k elements.
- At most n delete-min and n insertions.
  - Since those arrays contains n elements in total.

Overall, our algorithms takes $O(k \log k) + n \cdot O(\log k) = O(n \log k)$ time.

Special Exercise 8 Problem 4

Problem

Let S be a dynamic set of integers. At the beginning, S is empty. Then, new integers are added to it one by one, but never deleted. Let k be a fixed integer. Describe an algorithm which achieves the following guarantees:

- Space consumption $O(k)$
- `Insert(e)`: Insert a new element e into S, which takes at most $O(\log k)$ time.
- `Report-top-k`: Report the k largest integers in S.

Example

Suppose that $k = 3$, and the sequence of integers inserted is $83, 21, 66, 5, 24, 76, 92, 33, 43, \cdots$. Your algorithm must be keeping $\{83, 66, 24\}$ after the insertion of 24, $\{83, 66, 76\}$ after the insertion of 76, and $\{83, 76, 92\}$ after the insertion of 43.

Special Exercise 8 Problem 4

Solution 1

We maintain a binary heap with k elements, which obviously consumes $O(k)$ space.

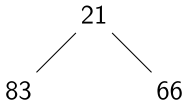
- First, perform k insertions to build a binary heap H rooted at r on the first inserted k elements of S, and each insertion takes at most $O(\log k)$ time.
- For a newly inserted integer e, compare it with the root r of H:
  - If $e > r$, replace r with e, and perform `root-fix` on H.
    - This takes $O(\log k)$ time.
  - Otherwise, ingore e.
- Then, at any moment, H contains the k largest integers of S.
  - $\texttt{Report-top-}k = \texttt{Report}(H)$.

Special Exercise 8 Problem 4

Example

Suppose that the sequence of integers inserted is:
$83, 21, 66, 5, 24, 76, 92, 33, 43, \cdots$, and $k = 3$.

First of all, build a binary heap H on the first inserted 3 elements $\{83, 21, 66\}$:

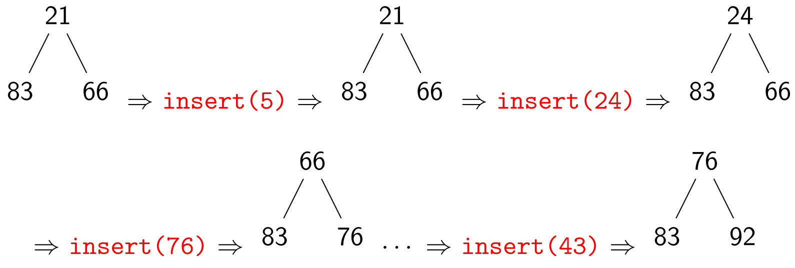


Special Exercise 8 Problem 4

Example

Suppose that the sequence of integers inserted is:
$83, 21, 66, 5, 24, 76, 92, 33, 43, \cdots$, and $k = 3$.

Next, we perform insertions one by one, and see what will happen on our binary heap H:



Special Exercise 8 Problem 4

Solution 2

We maintain an array A with length $2k$, which obviously consumes $O(k)$ space.

- First, append the first inserted k elements of S to A.
- Append the i-th ($i > k$) inserted integer of S to A. Once A is full, do the following:
  - Perform k-selection to find the k-th largest element of A, denoted by v.
  - Remove the elements which are smaller than v from A.
  - Rearrange A such that $A[1], A[2], \cdots, A[k]$ contains the k-largest element respectively.
  - `Report-top-`k.

Special Exercise 8 Problem 4

Example

Suppose that the sequence of integers inserted is:
$83, 21, 66, 5, 24, 76, 92, 33, 43, \cdots$, and $k = 3$.

First of all, creat an array A with length $2k = 6$.

A:

Then keep insertion one by one.

Special Exercise 8 Problem 4

Example

$S = \{83, 21, 66, 5, 24, 76, 92, 33, 43, \cdots\}$, $k = 3$.

$$\begin{array}{|c|c|c|c|c|c|}\hline 83 & & & & & \\ \hline\end{array} \Rightarrow \begin{array}{|c|c|c|c|c|c|}\hline 83 & 21 & & & & \\ \hline\end{array} \Rightarrow \begin{array}{|c|c|c|c|c|c|}\hline 83 & 21 & 66 & & & \\ \hline\end{array}$$

$$\Rightarrow \begin{array}{|c|c|c|c|c|c|}\hline 83 & 21 & 66 & 5 & & \\ \hline\end{array} \Rightarrow \begin{array}{|c|c|c|c|c|c|}\hline 83 & 21 & 66 & 5 & 24 & \\ \hline\end{array} \Rightarrow \begin{array}{|c|c|c|c|c|c|}\hline 83 & 21 & 66 & 5 & 24 & 76 \\ \hline\end{array}$$

A is full now. We perform *k*-selection to find the 3rd-largest integer, which is 66. Then remove the elements which are smaller than 66 from A:

$$\begin{array}{|c|c|c|c|c|c|}\hline 83 & & 66 & & & 76 \\ \hline\end{array} \Rightarrow \text{rearrange} \Rightarrow \begin{array}{|c|c|c|c|c|c|}\hline 83 & 66 & 76 & & & \\ \hline\end{array}$$

So the top-*k* (top-3) elements are $\{83, 66, 76\}$. We can continue insertion like this.

Cost Analysis

- Append the first inserted elements of S to A takes $O(k)$ time.
- Keep insertion, once A is full, we perform k-selection to report top-k, which takes $O(k)$ time.
- Remove the elements and rearrange A takes $O(k)$ time.

Overall, our algorithm takes $O(k)$ time. Charge these costs to the k insertions indicated below, each insertion bears $O(1)$ time, and each insertion is only charged once.