Exercises on the Growth of Functions

CSCI2100 Tutorial 2

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Adapted from the slides of the previous offerings of the course

Introduction

Last week, we have learned two different ways to decide whether one function $f(n)$ grows faster than another $g(n)$:

- The first one achieves the purpose by finding appropriate "constants c_1, c_2".
- The second is by inspecting the ratio $\frac{f(n)}{g(n)}$ as $n \to \infty$.

In this tutorial, we will apply both methods through some exercises.

Exercise 1

Let $f(n) = 10n + 5$ and $g(n) = n^2$. Prove $f(n) = O(g(n))$ and $g(n) \neq O(f(n))$.

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Let $f(n) = 10n + 5$ and $g(n) = n^2$. Prove $f(n) = O(g(n))$ and $g(n) \neq O(f(n))$.

Direction 1: Constant Finding

$f(n) = O(g(n))$, if there exist two positive constants c_1 and c_2 such that $f(n) \leq c_1 \cdot g(n)$ holds for all $n \geq c_2$.

Exercise 1

Let $f(n) = 10n + 5$ and $g(n) = n^2$. Prove $f(n) = O(g(n))$ and $g(n) \neq O(f(n))$.

Direction 1: Constant Finding

Proof of $f(n) = O(g(n))$

Our mission is to find c_1, c_2 to make $f(n) \leq c_1 \cdot g(n)$ hold for all $n \geq c_2$. Remember: we do not need to find the smallest c_1, c_2; instead, it suffices to obtain any c_1, c_2 that can do the job. Indeed, we will often go for some "easy" selections that can simplify derivation.

Exercise 1

Let $f(n) = 10n + 5$ and $g(n) = n^2$. Prove $f(n) = O(g(n))$ and $g(n) \neq O(f(n))$.

Direction 1: Constant Finding

Proof of $f(n) = O(g(n))$

$$
\begin{aligned}
& 10n + 5 \leq c_1 \cdot n^2 \\
\Leftrightarrow \quad & 5(2n+1) \leq c_1 \cdot n^2 \qquad (\text{let } c_1 = 5) \\
\Leftrightarrow \quad & 2n + 1 \leq n^2 \\
\Leftrightarrow \quad & 2 \leq (n-1)^2 \\
\Leftarrow \quad & 3 \leq n
\end{aligned}
$$

Hence, it suffices to set $c_2 = 3$.

Exercise 1

Let $f(n) = 10n + 5$ and $g(n) = n^2$. Prove $f(n) = O(g(n))$ and $g(n) \neq O(f(n))$.

Direction 1: Constant Finding

Proof of $g(n) \neq O(f(n))$

Let us prove this by contradiction. Suppose, on the contrary, that $g(n) = O(f(n))$. This means the existence of constants c_1, c_2 such that, we have for all $n \geq c_2$

$$
\begin{aligned}
& n^2 \leq c_1 \cdot (10n + 5) \\
\Rightarrow \quad & n^2 \leq c_1 \cdot 20n \\
\Leftrightarrow \quad & n \leq 20c_1
\end{aligned}
$$

which cannot always hold for all $n \geq c_2$. This completes the proof.

Exercise 1

Let $f(n) = 10n + 5$ and $g(n) = n^2$. Prove $f(n) = O(g(n))$ and $g(n) \neq O(f(n))$.

Direction 2: Inspecting $\lim_{n\to\infty} \frac{f(n)}{g(n)}$

Proof of $f(n) = O(g(n))$

$$\lim_{n\to\infty} \frac{10n+5}{n^2} = \lim_{n\to\infty} \frac{10+5/n}{n} = 0.$$

Hence, $f(n) = O(g(n))$.

Exercise 1

Let $f(n) = 10n + 5$ and $g(n) = n^2$. Prove $f(n) = O(g(n))$ and $g(n) \neq O(f(n))$.

Direction 2: Inspecting $\lim_{n\to\infty} \frac{f(n)}{g(n)}$

Proof of $g(n) \neq O(f(n))$

$$\lim_{n\to\infty} \frac{n^2}{10n+5} = \infty.$$

Hence, $g(n) \neq O(f(n))$.

Exercise 2

Let $f(n) = 5\log_2 n$ and $g(n) = \sqrt{n}$. Prove $f(n) = O(g(n))$ and $g(n) \neq O(f(n))$.

Direction 1: Constant Finding

Proof of $f(n) = O(g(n))$

Setting $c_1 = 5$, we want:

$$\begin{aligned} & 5 \log_2 n \leq 5 \cdot \sqrt{n} \\ \Leftrightarrow \quad & \log_2 n \leq \sqrt{n} \end{aligned}$$

Hence, it suffices to set $c_2 = 64$.

Direction 1: Constant Finding

Proof of $g(n) \neq O(f(n))$

We prove this by contradiction. Suppose that $g(n) = O(f(n))$. It implies that there exist constants c_1, c_2 such that for all $n \geq c_2$, we have

$$\begin{aligned} & \sqrt{n} \leq c_1 \cdot 5 \cdot \log_2 n \\ \Leftrightarrow \quad & \frac{\sqrt{n}}{\log_2 n} \leq 5c_1 \end{aligned}$$

which cannot always hold for all $n \geq c_2$. This completes the proof.

Direction 2: Inspecting $\lim_{n\to\infty} \frac{f(n)}{g(n)}$

Proof of $f(n) = O(g(n))$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{5\log_2 n}{\sqrt{n}} = 0.$$

Thus, we have $f(n) = O(g(n))$.

Proof of $g(n) \neq O(f(n))$.

$$\lim_{n\to\infty} \frac{g(n)}{f(n)} = \lim_{n\to\infty} \frac{\sqrt{n}}{5\log_2 n} = \infty.$$

Hence, $g(n) \neq O(f(n))$.

Exercise 3

Given that $f(n) = O(g(n))$, prove $\sqrt{f(n)} = O(\sqrt{g(n)})$.

Direction 1: Constant Finding

Since $f(n) = O(g(n))$ implies the existence of constants c_1 and c_2 such that $f(n) \leq c_1 \cdot g(n)$ holds for all $n \geq c_2$.

Thus:

$$\sqrt{f(n)} \leq \sqrt{c_1 \cdot g(n)} = \sqrt{c_1} \cdot \sqrt{g(n)}$$

holds for all $n \geq c_2$.

Therefore, by setting $c_1' = \sqrt{c_1}$ and $c_2' = c_2$, we have $\sqrt{f(n)} = O(\sqrt{g(n)})$.

Direction 2: Inspecting $\lim_{n\to\infty} \frac{f(n)}{g(n)}$

Since $f(n) = O(g(n))$, we have $\lim_{n\to\infty} \frac{f(n)}{g(n)} = c$, where c is some constant.

The above implies that:

$$\lim_{n\to\infty} \frac{\sqrt{f(n)}}{\sqrt{g(n)}} = \lim_{n\to\infty} \sqrt{\frac{f(n)}{g(n)}} = \sqrt{\lim_{n\to\infty} \frac{f(n)}{g(n)}} = \sqrt{c}$$

Therefore, $\sqrt{f(n)} = O(\sqrt{g(n)})$.

Note that in this proof we assume that $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ exists, the same assumption holds in the following exercise.

Exercise 4

Consider functions of n: $f_1(n)$, $f_2(n)$, $g_1(n)$ and $g_2(n)$ such that:

$$f_1(n) = O(g_1(n)) \text{ and } f_2(n) = O(g_2(n))$$

Prove $f_1(n) + f_2(n) = O(g_1(n) + g_2(n))$.

Direction 1: Constant Finding

Since $f_1(n) = O(g_1(n))$, there exist constants c_1 and c_2 such that $f_1(n) \le c_1 \cdot g_1(n)$ holds for all $n \ge c_2$.

Similarly, $f_2(n) = O(g_2(n))$ implies the existence of constants c_1' and c_2' such that $f_2(n) \le c_1' \cdot g_2(n)$ holds for all $n \ge c_2'$.

Thus:

$$f_1(n) + f_2(n) \le c_1 \cdot g_1(n) + c_1' \cdot g_2(n) \le \max\{c_1, c_1'\} \cdot (g_1(n) + g_2(n))$$

for all $n \ge \max\{c_2, c_2'\}$.

Therefore, $f_1(n) + f_2(n) = O(g_1(n) + g_2(n))$.

Direction 2: Inspecting $\lim_{n\to\infty} \frac{f(n)}{g(n)}$

Since $f_1(n) = O(g_1(n))$, we have $\lim_{n\to\infty} \frac{f_1(n)}{g_1(n)} = c$ for some constant c.

Similarly, $f_2(n) = O(g_2(n))$ indicates $\lim_{n\to\infty} \frac{f_2(n)}{g_2(n)} = c'$ for some constant c'.

This leads to:

$$\begin{aligned} \lim_{n\to\infty} \frac{f_1(n) + f_2(n)}{g_1(n) + g_2(n)} &= \lim_{n\to\infty} \frac{f_1(n)}{g_1(n) + g_2(n)} + \lim_{n\to\infty} \frac{f_2(n)}{g_1(n) + g_2(n)} \\ &\leq \lim_{n\to\infty} \frac{f_1(n)}{g_1(n)} + \lim_{n\to\infty} \frac{f_2(n)}{g_2(n)} \\ &\leq c + c'. \end{aligned}$$

Therefore, $f_1(n) + f_2(n) = O(g_1(n) + g_2(n))$.