More Examples and Applications on AVL Tree

CSCI2100 Tutorial 11

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Adapted from the slides of the previous offerings of the course

Recall in lectures we studied the AVL tree, which is one type of self-balancing binary search tree. The aim was to store a set of integers S supporting the following operations:

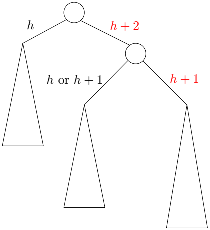
- A predecessor query: given an integer q, find its predecessor in S;
- Insertion: add a new integer to S; and
- Deletion: remove an integer from S.

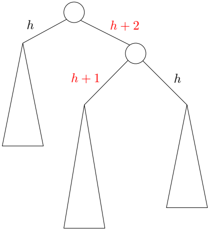
We want all of these operations to run in $O(\log n)$ (where n is the number of integers in S) in the worst case. If we were to attempt to accomplish this using a BST, we must ensure it is balanced after every operation, and the AVL tree presents one method of doing so.

Rebalancing

We know that a tree is balanced as long as the height of its subtrees differ by at most 1, and that insertion and deletion can only cause a 2-level imbanace (where the heights differ by 2).

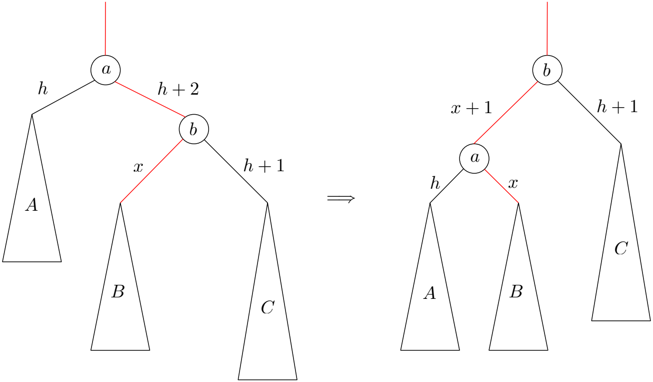
In lectures we explored the Left-Left and Left-Right cases in detail, so here we will look at Right-Right and Right-Left:





Right-Right

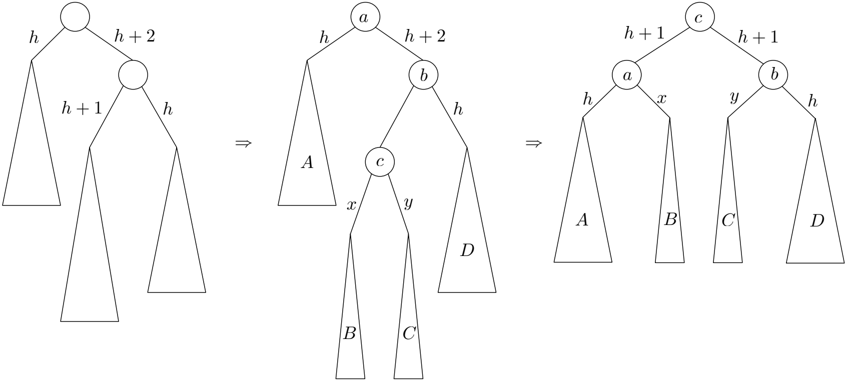
Similar to Left-Left, fix by a rotation:



Note that $x = h$ or $h + 1$, and the ordering from left to right of A, a, B, b, C is preserved after rotation.

Right-Left

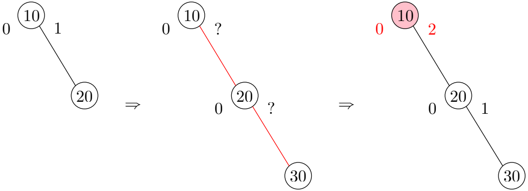
Similar to Left-Right, fix by a double rotation:



Note that x and y must be h or $h - 1$. Futhermore at least one of them must be h.

Insertion

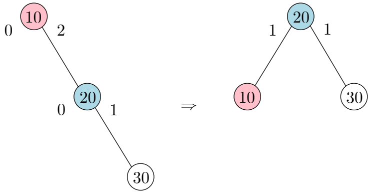
Let's see these in action with some concrete examples involving insertion and deletion. Suppose we start with an empty tree and add 10, 20 and 30. Inserting 30 yields:



We first traverse from root-to-leaf and add a node with key 30. The height of the subtrees along this path are now invalidated, so we traverse back up to the root and recalculate at each node. When we get to node 10, we find that we have an imbalance, in this case of type Right-Right.

Insertion

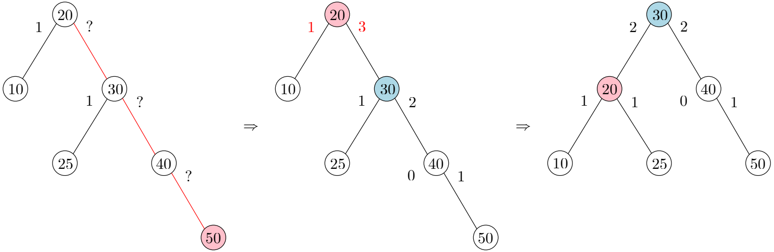
We fix via a rotation according to the Right-Right case:



One should check that at the end the tree is balanced, and satisfies the binary search tree property!

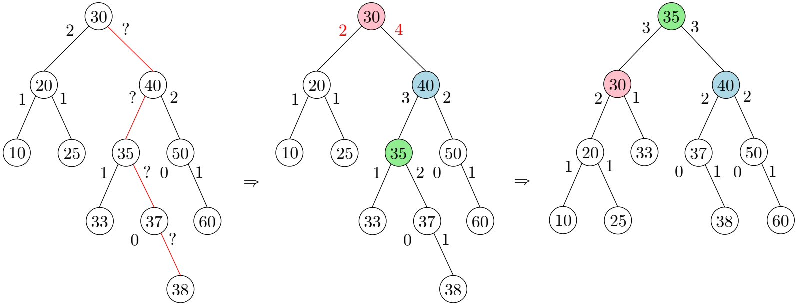
Insertion

Suppose we then add 25, 40 and 50. Upon inserting 50, we will find the need for another Right-Right rebalancing:



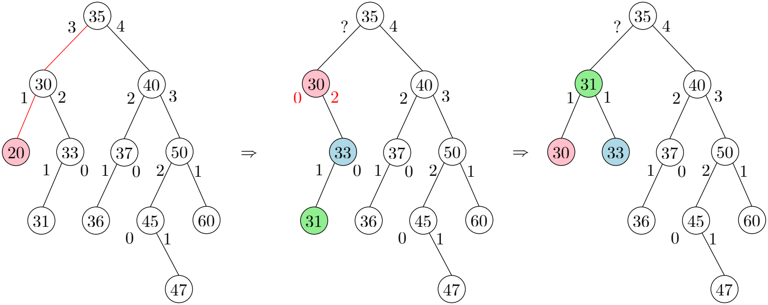
Insertion

To get a Right-Left case, let's add 35, 33, 37, 60 and 38 (in this exact order). Upon inserting 38 we will find:



Deletion

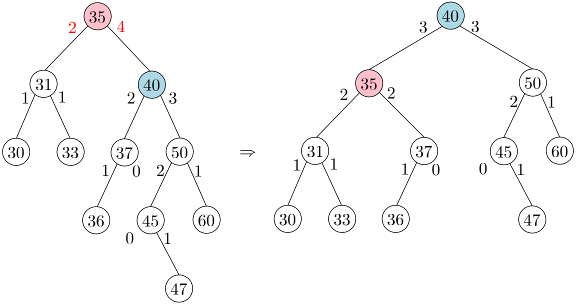
Let's look at an example of a deletion as well: suppose that some sequence of operations later we arrive at the following and want to delete the key 20:



After identifying and deleting the appropriate node (more on this later), we again need to traverse back up to the root and rebalance. Here we run into a Right-Left case.

Deletion

. . . and we are actually still not done because there is another Right-Right rebalance we need to do:



Remember that at most one rebalance is needed on insert; but deletion may require more than one!

Special Exercise 10 Problem 5

Problem

Let S be a dynamic set of integers, and $n = |S|$. Describe a data structure to support the following operations on S with the required performance guarantees:

- Insert a new element to S in $O(\log n)$ time.
- Delete an element from S in $O(\log n)$ time.
- Report the k smallest elements of S in $O(k)$ time, for any k satisfying $1 \leq k \leq n$.

Your data structure must consume $O(n)$ space at all times.

Special Exercise 10 Problem 5

Solution

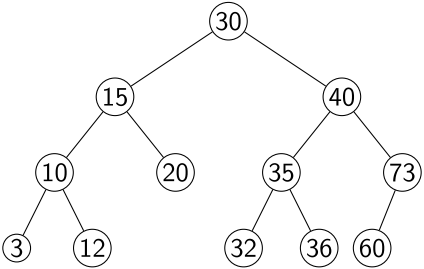
We maintain a binary search tree T over S, which consumes $O(n)$ space at all times.

- For insertion, insert a new element into T, rebalance T and identify the smallest element e in T ($O(\log n)$ time).
- For deletion, remove an element from T, rebalance T and identify the smallest element e in T ($O(\log n)$ time).
- For reporting the k smallest elements, start from e and repeat the following until we find the k elements:
  - Starting from current node, find the next node which stores the successor s of the current node, and report s.

Special Exercise 10 Problem 5

Example

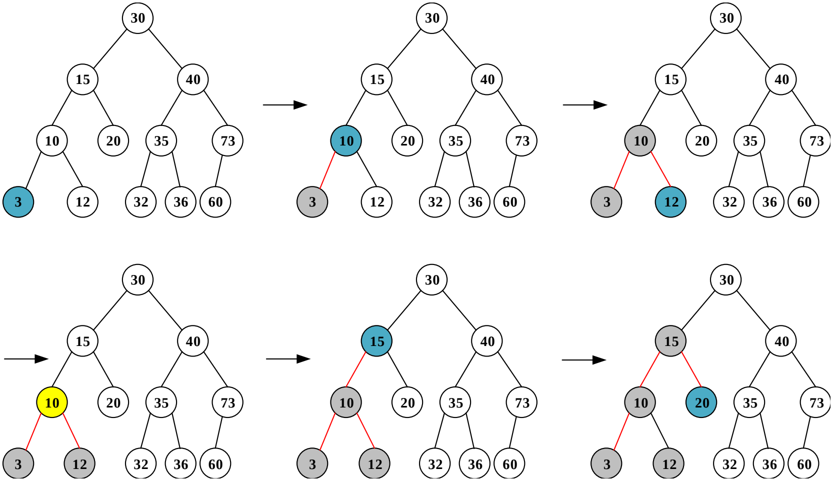
Suppose that $k = 5$ and after a sequence of insertions and deletions, we have the following BST:



Special Exercise 10 Problem 5

Example

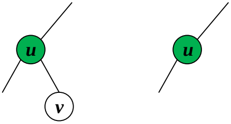
Then we can report the 5 smallest elements as follows:



Cost analysis

Lemma: During the process of traversing the k smallest elements, any node can be visited at most twice.

Proof: In our algorithm, We traverse the k elements in a bottom-up manner. Therefore, there are only two cases for visiting a node u:



Case 1 Case 2

Case 1: u has a right child. After visiting u, we have to visit its child node v, and then go back to visit u again. Hence, u was visited twice.

Case 2: u does not have a right child. After visiting u, we directly go up to visit next node and never come back to u again. Hence, u is only visited once. □

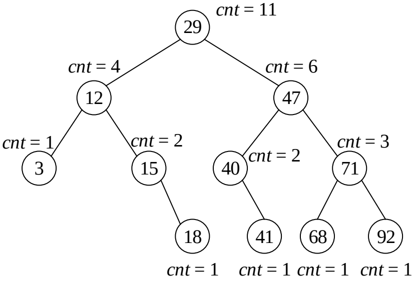
Cost analysis

Since visiting a node once takes $O(1)$ time, and based on the lemma we know that a node can be visited at most twice.

Hence, the time for reporting k smallest elements is bounded by $2 \cdot k \cdot O(1) = O(k)$.

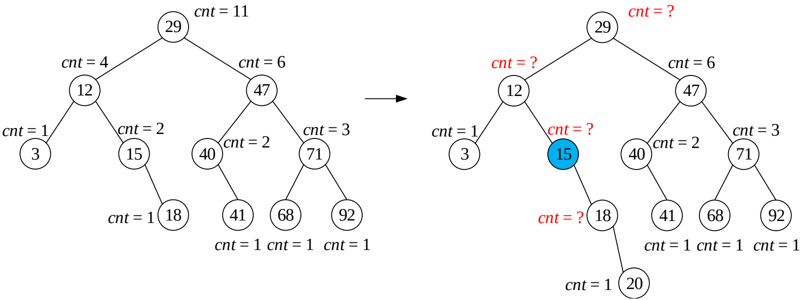
BST with counter – dynamic version

In tutorial 10, we have introduced range count problem and solved it by augmenting a balanced BST by storing the number of nodes in the subtree of u in each node u. For example:



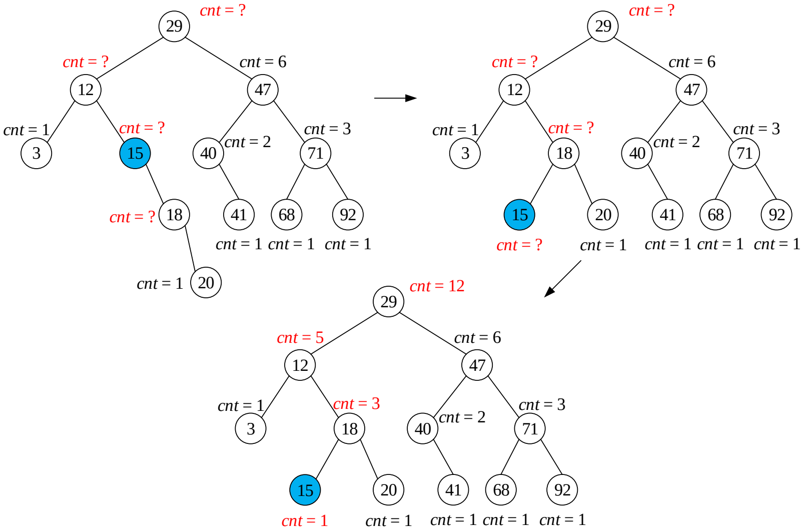
When we perform a insertion or deletion, we need to rebalance the BST as well as updating the counter values of some nodes.

Suppose that we insert 20 into the following BST:



The node with key 15 becomes imbalanced, and the counter values of the nodes along the path we traversed when inserting 20 are now invalidated.

We first rebalance the BST via a rotation(Right-Right case), and then recalculate the counter values of the afore mentioned nodes.



Recall that when performing insertion or deletion, we essentially descend a root-to-leaf path, only the counter values of the nodes on this path need to be updated.

Therefore, we can update the counter values of these nodes in the bottom-up order, which follows the same idea with the updating of the subtree height values of these nodes.

Count of Smaller Numbers Problem

Problem

Given an array A with n integers. Describe an algorithm to produce a new array B, such that $B[i]$ $(1 \leq i \leq n)$ is the number of elements in A that (i) are smaller than $A[i]$, and (ii) are to the right of $A[i]$.

Example

A:	29	12	47	40	71	15	3	41	18	11	92	68

To the right of $A[8] = 41$, there are 2 smaller elements, then $B[8] = 2$;
To the right of $A[5] = 71$, there are 6 smaller elements, then $B[5] = 6$;
$\cdots\cdots$
Hence, the output array B should be:

B:	5	2	6	4	6	2	0	2	1	0	1	0

Solution

Clearly, $B[n] = 0$. We maintain a balanced BST T with counters. At the beginning, T contains only $A[n]$.

Then, for $i = n - 1$ downto 1, do the following:

- Perform a range count, i.e., find the number of integers in T which are in the range of $(-\infty, A[i]]$. Denote the result by c.
- Set $B[i] = c$.
- Insert $A[i]$ to T.

Example

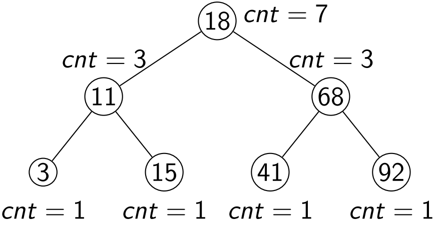
i

A:

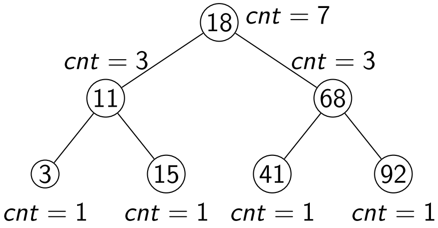
29	12	47	40	71	15	3	41	18	11	92	68

$n = 12$

Suppose that after a sequence of operations, $i = 5$, then our BST T constructed on the elements from $A[6]$ to $A[12]$ should be:



Example



Since $A[5] = 71$, perform a range count to find the number of integers in the above BST which are in the range of $(3, 71]$. This gives $c = 6$. Hence, $B[5] = 6$.

Cost Analysis

- For each element in $A[i]$, we perform an insertion and a range count, which takes at most $O(\log n)$ time.
- We have n elements in total.

Hence, the whole algorithm terminates in $O(n \log n)$ time.