Applications of the Binary Search Tree

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Adapted from the slides of the previous offerings of the course

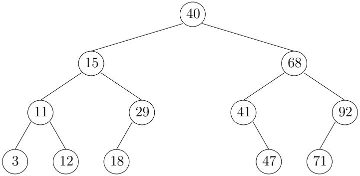
Recall

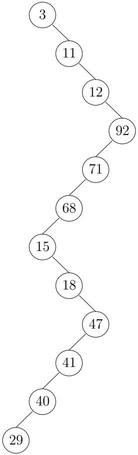
A binary search tree (BST) on a set S of n integers is a binary tree T satisfying all the following requirements:

- T has n nodes.
- Each node u in T stores a distinct integer in S, which is called the key of u.
- For every internal u, it holds that:
  - The key of u is larger than all the keys in the left subtree of u.
  - The key of u is smaller than all the keys in the right subtree of u.

Example

Two possible BSTs on $S = \{3, 11, 12, 15, 18, 29, 40, 41, 47, 68, 71, 92\}$:



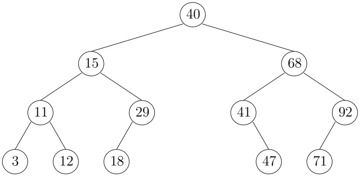


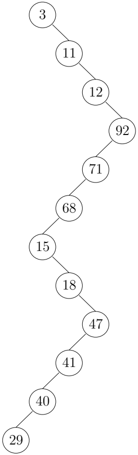
Recall

A binary tree T is balanced if the following holds on every internal node u of T:

- The height of the left subtree of u differs from that of the right subtree of u by at most 1.

Example





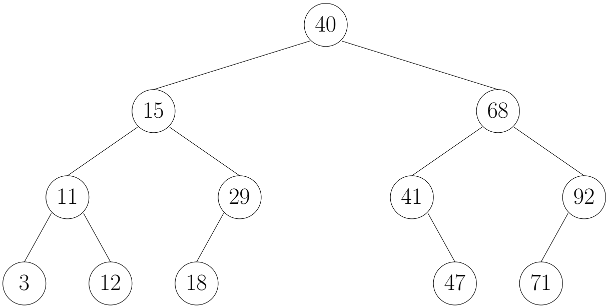
The BST on the left is balanced, while the one on the right is not.

Predecessor Query

Let S be a set of integers. A predecessor query for a given integer q is to find its predecessor in S, which is the largest integer in S that does not exceed q.

Example

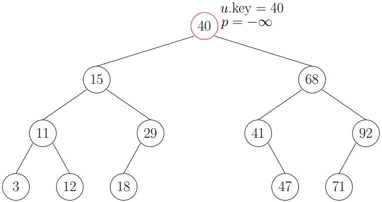
Suppose that $S = \{3, 11, 12, 15, 18, 29, 40, 41, 47, 68, 71, 92\}$ and we have a balanced BST T on S:



We want to find the predecessor of $q = 42$ in S.

Example

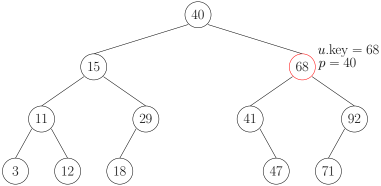
Predecessor query for $q = 42$:



- Initialize $p = -\infty$.
- Initialize $u \leftarrow$ the root of T.
- Now u.key $= 40$ and $p = -\infty$.
- Since u.key $< q$, the predecessor of q must be either u or some node in the right subtree of u.
- Set $p = 40$ and $u \leftarrow$ the right child of u.

Example

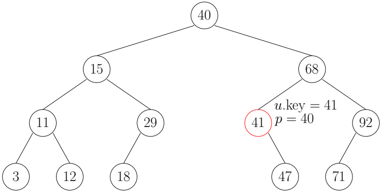
Predecessor query for $q = 42$:



- Since u.key $> q$, the predecessor of q must be either p or some node in the left subtree of u.
- Set $u \leftarrow$ the left child of u.

Example

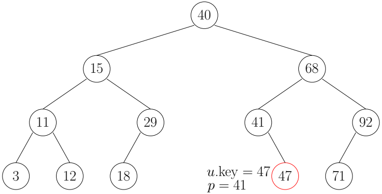
Predecessor query for $q = 42$:



- Since u.key $< q$, the predecessor of q must be either u or some node in the right subtree of u.
- Set $p = 41$ and $u \leftarrow$ the right child of u.

Example

Predecessor query for $q = 42$:



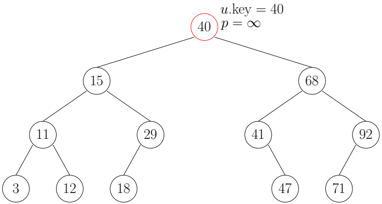
- Since u.key $> q$, the predecessor of q must be either p or some node in the left subtree of u.
- Set $u \leftarrow$ the left child of u.
- Since u is nil now, return $p = 41$ as the predecessor of q in S.

Successor Query

Let S be a set of integers. A successor query for a given integer q is to find its successor in S, which is the smallest integer in S that is no smaller than q.

Example

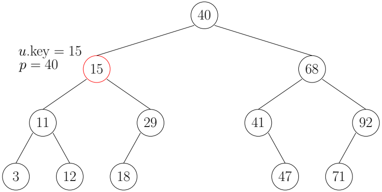
Successor query for $q = 17$ on S:



- Initialize $p = \infty$.
- Initialize $u \leftarrow$ the root of T.
- Now u.key $= 40$ and $p = \infty$.
- Since u.key $> q$, the successor of q must be either u or some node in the left subtree of u.
- Set $p = 40$ and $u \leftarrow$ the left child of u.

Example

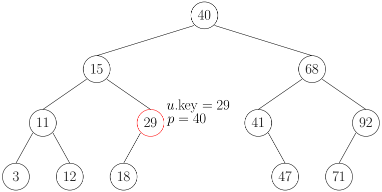
Successor query for $q = 17$ on S:



- Since u.key $< q$, the successor of q must be either p or some node in the right subtree of u.
- Set $u \leftarrow$ the right child of u.

Example

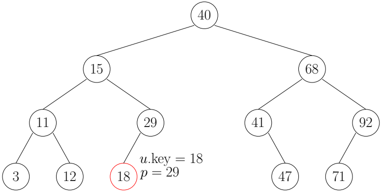
Successor query for $q = 17$ on S:



- Since u.key $> q$, the successor of q must be either u or some node in the left subtree of u.
- Set $p = 29$ and $u \leftarrow$ the left child of u.

Example

Successor query for $q = 17$ on S:



- Since u.key $> q$, the successor of q must be either u or some node in the left subtree of u.
- Set $p = 18$ and $u \leftarrow$ the left child of u.
- Since u is nil now, return $p = 18$ as the successor of q in S.

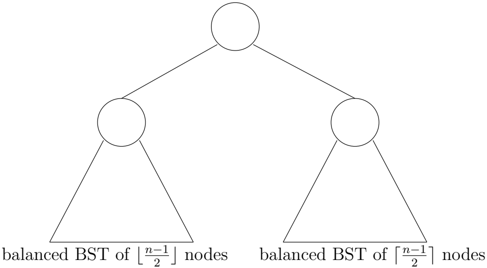
Construction of a Balanced BST

In the following, we will discuss how to construct a balanced BST T on a given sorted set S of n integers in $O(n)$ time.

Construction of a Balanced BST

- **Observation 1:** The subtree of any node in a balanced BST is also a balanced BST.

- **Observation 2:** A BST of n nodes constructed by the following form:



is a balanced BST(think: why?).

Construction of a Balanced BST

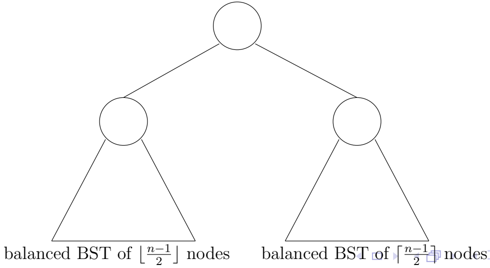
Assume that the S of n integers is stored in an array A, the array is sorted.

A balanced BST on S can be constructed as follows:

- Base Case:
  - If $n = 0$, return nil.
  - If $n = 1$, create a node u with key $A[1]$ and return the pointer of u as the root of a balanced BST on A.

Construction of a Balanced BST

- Inductive Case:
  - Pick the median of A (i.e., $A[\lfloor \frac{n+1}{2} \rfloor]$) and create a node u for it.
  - Recursively construct a balanced BST on the portion of A positioned before the median, and set its root as the left child of u.
  - Recursively construct a balanced BST on the portion of A positioned after the median, and set its root as the right child of u.
  - Return the pointer of u.



Construction of a Balanced BST

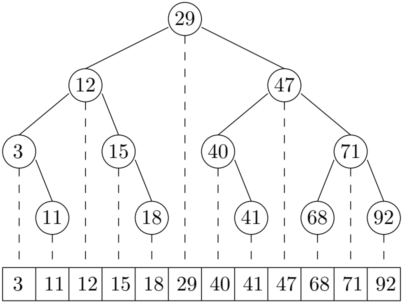
Let $f(n)$ be the maximum running time for constructing a balanced BST from an array of length n. Without loss of generality, suppose that n is a power of 2. We have:

$$f(1) = O(1)$$
$$f(n) = O(1) + 2 \cdot f(n/2)$$

Solving the recurrence gives $f(n) = O(n)$.

Example

Let us construct a balanced BST T on a sorted set $S = \{3, 11, 12, 15, 18, 29, 40, 41, 47, 68, 71, 92\}$ by the above algorithm. Suppose that S is stored in an array A of length 12.



Range Count Problem

Let S be a set of n integers. Given two integers a and b such that $a \leq b$. Find the number of integers in S which are in the range of $[a, b]$.

In the following, we will discuss how to augment a balanced BST on S to achieve:

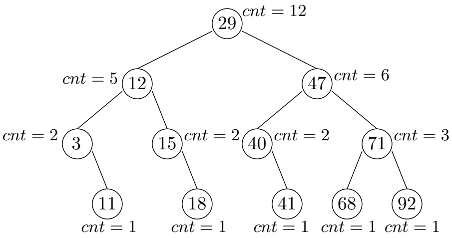
- $O(n)$ space consumption,
- $O(\log n)$ time for each query.

Range Count Problem

Augment a balanced BST T on S by storing one additional information in each node u that is:

- the number of nodes in the subtree of u.

For example,

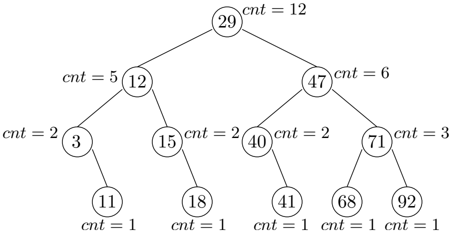


Range Count Problem

Define a concept first.

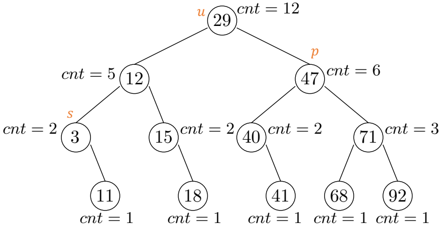
- Lowest Common Ancestor: Let t be the root. The lowest common ancestor of nodes v_1 and v_2 is the lowest node that is on both of the paths $P(t, v_1)$ and $P(t, v_2)$.

For example, the lowest common ancestor of node with key 3 and node with key 15 is the node with key 12.



Range Count Problem

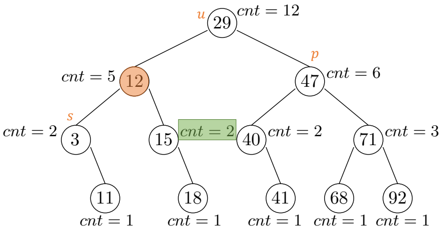
For a range $[2, 48]$, let s be the successor of 2, p the predecessor of 48 and u the lowest common ancestor of s and p.
Initialize a count $c = 1$ (since u is within the range)



Range Count Problem

Traverse the path from u's left child to s.
For every node v being visited, if v.key ≥ 2:

- c $+= 1$
- c $+=$ the counter of v's right child

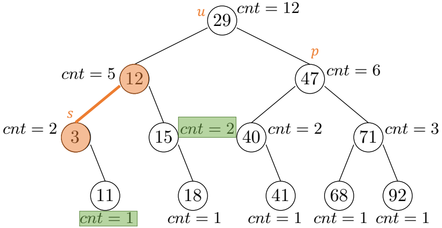


C is incremented by $1 + 2$.

Range Count Problem

Traverse the path from u's left child to s.
For every node v being visited, if v.key ≥ 2:

- c $+= 1$
- c $+=$ the counter of v's right child

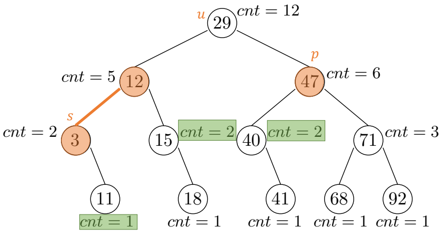


C is incremented by $1 + 1$.

Range Count Problem

Traverse the path from u's right child to p.
For every node v being visited, if v.key $\leq$ 48:

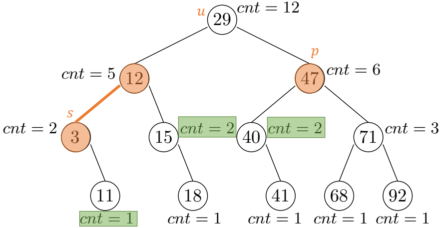
- c $+=$ 1
- c $+=$ the counter of v's left child



C is incremented by $1 + 2$. Finally, c becomes 9.

Range Count Problem

We walked through two paths, at most $\log_2 n$ nodes in each path.
For each node visited, we perform constant-time operations, which takes $O(1)$.
Time complexity: $O(\log n)$



Range Count Problem

A simpler solution without using a binary search tree

- Use binary search algorithm to find the successor s of a
- Use binary search algorithm to find the predecessor p of b
- Let l be the index of s, then let u be the index p
- Return $u - l + 1$

The above algorithm uses two binary search, the time complexity is $O(\log n)$.

Range Count Problem

Why don't we just use the simpler solution?
In practice, we may need to update (insert or delete) the elements in S.
Simpler Solution:

- Need to sort S after each update.
- Cost for each update: $O(n \log n)$

Solution with BST:

- Need to insert or delete a node in the BST.
- Cost for each update: $O(\log n)$

That's why we prefer the BST solution in practice.