Basic Concepts and Properties of Trees

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This lecture provides a formal definition of trees, which constitute an important approach to organize data in computer science. We will also prove some basic properties of trees that will be useful later in the course.

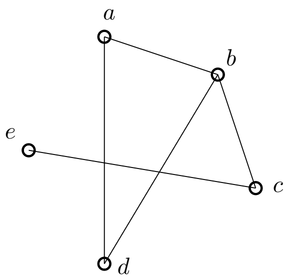
Undirected Graphs

An undirected simple graph is a pair of (V, E) where:

- V is a set of elements, each of which called a node.
- E is a set of **unordered pairs** $\{u, v\}$ such that u and v are distinct nodes.

A node may also be called a vertex. We will refer to V as the vertex set or the node set of the graph, and E the edge set.

Example



This is a graph (V, E) where

- $V = \{a, b, c, d, e\}$
- $E = \{\{a, b\}, \{b, c\}, \{a, d\}, \{b, d\}, \{c, e\}\}$.
  - The number of edges equals $|E| = 5$.

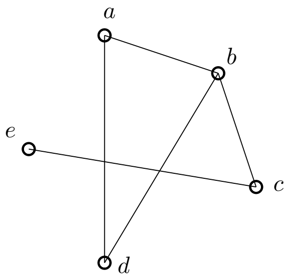
Paths and Cycles

Let $G = (V, E)$ be an undirected simple graph. A path in G is a sequence of nodes $(v_1, v_2, ..., v_k)$ such that

- For every $i \in [1, k-1]$, there is an edge between v_i and v_{i+1}.

A cycle in G is a path $(v_1, v_2, ..., v_k)$ such that $k \geq 4$ and $v_1 = v_k$.

Example

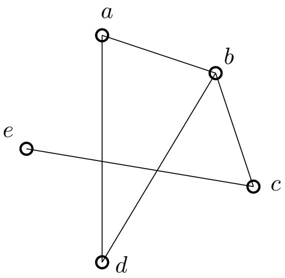


(a, b, d, a) is a cycle, whereas (a, b, c, e) is a path but not a cycle.

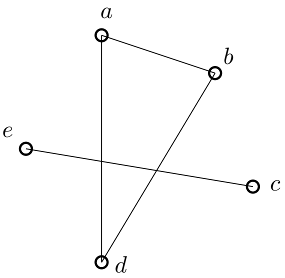
Connected Graphs

An undirected graph $G = (V, E)$ is connected if, for any two distinct vertices u and v, G has a path from u to v.

Example



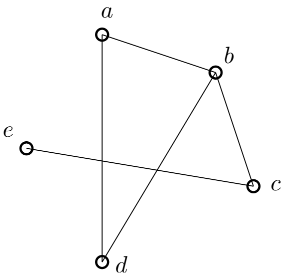
Connected



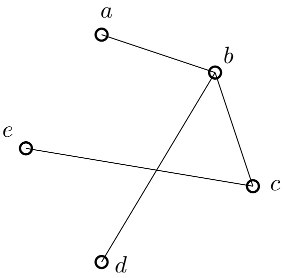
Not connected

Trees

A tree is a connected undirected graph contains no cycles.



Not a tree



A tree

A Property

Lemma: A tree with n nodes has $n - 1$ edges.

The proof will be left to you as an exercise.

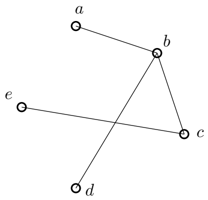
Rooting a Tree

Given any tree T and an arbitrary node r, we can allocate a level to each node as follows:

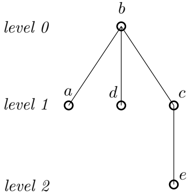
- r is the root of T—this is level 0 of the tree.
- All the nodes that are 1 edge away from r constitute level 1 of the tree.
- All the nodes that are 2 edges away from r constitute level 2 of the tree.
- And so on.

The number of levels is called the height of the tree. We say that T has been rooted once a root has been designated.

Example

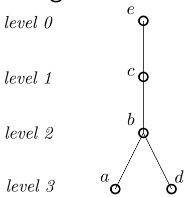


Rooting the tree at b



Height 3

Rooting the tree at e



Height 4

Concepts on Rooted Trees—Parents and Children

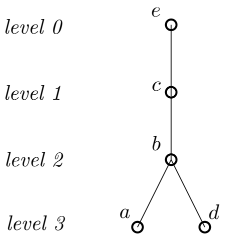
Consider a tree T that has been rooted.

Let u and v be two nodes in T. We say that u is the parent of v if:

- v is at the level immediately below u, and
- There is an edge between u and v.

Accordingly, we say that v is a child of u.

Example



Node b is the parent of two child nodes: a, d.
Node e is the parent of c, which is in turn the parent of b.

Concepts on Rooted Trees—Ancestors and Descendants

Consider a rooted tree T.

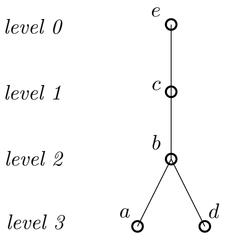
Let u and v be two nodes in T. We say that u is an ancestor of v if one of the following holds:

- $u = v$,
- u is the parent of v, or
- u is the parent of an ancestor of v.

Accordingly, we say that v is a descendant of u.

In particular, if $u \neq v$, we say that u is a proper ancestor of v, and likewise, v is a proper descendant of u.

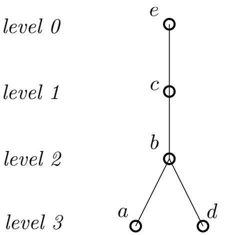
Example



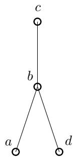
Node b is an ancestor of b, a and d.
Node c is an ancestor of c, b, a, and d.
Node c is a proper ancestor of b, a, d.

Concepts on Rooted Trees—Subtrees

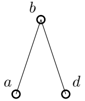
Let u be a node in a rooted tree T. The subtree of u is the part of T that is "at or below" u.



Tree



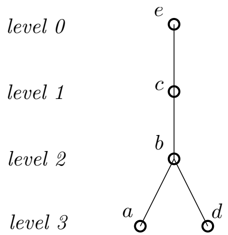
Subtree of c



Subtree of b

Concepts on Rooted Trees—Internal and Leaf Nodes

In a rooted tree, a node is a leaf node if it has no children; otherwise, it is an internal node.



Internal nodes: e, c, and b. Leaf nodes: a and d.

A Property

Lemma: Let T be a rooted tree where every internal node has at least 2 child nodes. If m is the number of leaf nodes, then the number of internal nodes is at most $m - 1$.

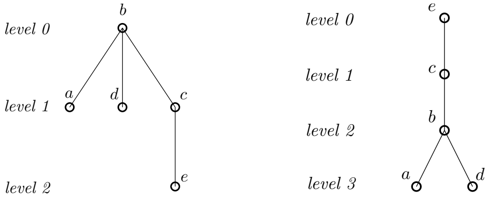
Proof: Consider the tree as a schedule for a tournament as follows. The competing teams are initially placed at the leaf nodes. Each internal node v represents a match among the teams at the child nodes, such that only the winning team advances to the match at the parent of v. The team that wins the match at the root is the champion.

Each match eliminates at least one team. There are at most $m - 1$ teams to eliminate before the champion is determined. Hence, there can be at most $m - 1$ matches (i.e., nodes). □

Concepts on Rooted Trees—k-Ary and Binary

A *k*-ary tree is a rooted tree where every internal node has at most k child nodes.

A 2-ary tree is called a binary tree.



3-ary Binary

Concepts on a Binary Tree—Left and Right

A binary tree is left-right labeled if

- Every node v—except the root—has been designated either as a left or right node of its parent.
- Every internal node has at most one left child, and at most one right child.

> Throughout this course, we will discuss only binary trees that have been left-right labeled. Because of this, by a "binary tree", we always refer to a left-right labeled one.

Concepts on a Binary Tree—Left and Right

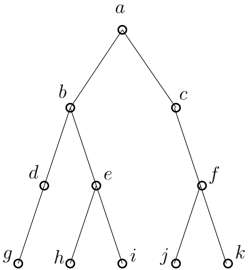
A (left-right labeled) binary tree implies an ordering among the nodes at the same level.

Let u, v be nodes at the same level with parents p_u and p_v, respectively. We say that u is on the left of v if either of the following holds:

- $p_u = p_v$, and u is the left child (implying that v is the right child);
- $p_u \neq p_v$, and p_u is on the left of p_v.

Accordingly, we say that v is on the right of u.

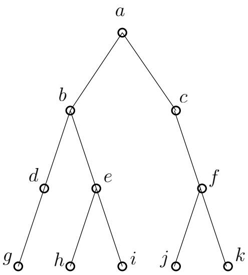
Example



At Level 1, b is on the left of c.
At Level 2, the nodes from left to right are d, e, and f.
At Level 3, the nodes from left to right are g, h, i, j, and k.

Concepts on a Binary Tree—Full Level

Consider a binary tree with height h. Its Level ℓ ($0 \le \ell \le h - 1$) is full if it contains 2^ℓ nodes.



Levels 0 and 1 are full, but Levels 2 and 3 are not.

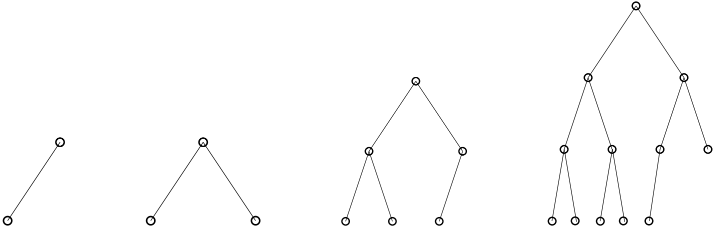
Concepts on a Binary Tree—Complete Binary Tree

A binary tree of height h is complete if:

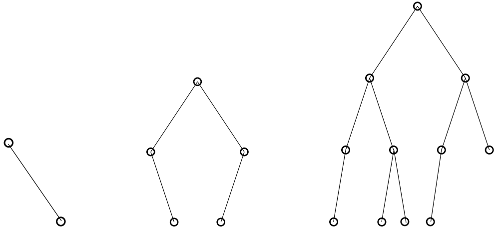
- Levels 0, 1, ..., $h - 2$ are all full (i.e., the only possible exception is the bottom level).
- At Level $h - 1$, the leaf nodes are "as far left as possible".
  - This means that if you were to add a leaf node v at Level $h - 1$, v would need to be on the right of all the existing leaf nodes.

Example

Complete binary trees:



Not complete binary trees:



A Property

Lemma: A complete binary tree with $n \geq 2$ nodes has height $O(\log n)$.

Proof: Let h be the height of the binary tree. As Levels $0, 1, ..., h-2$ are full, we know that

$$\begin{aligned} 2^0 + 2^1 + ... + 2^{h-2} &\leq n \\ \Rightarrow \quad 2^{h-1} - 1 &\leq n \\ \Rightarrow \quad h &\leq 1 + \log_2(n+1) = O(\log n). \end{aligned}$$

□