Topological Sort on a DAG

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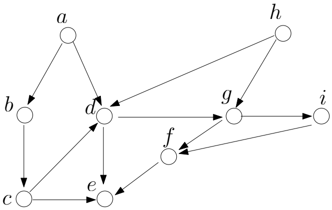
As mentioned earlier, depth first search (DFS) algorithm is surprisingly powerful. Indeed, we have already used it to detect efficiently whether a directed graph contains any cycle. In this lecture, we will use it to settle another classic problem—called topological sort—in linear time.

Topological Order

Let $G = (V, E)$ be a directed acyclic graph (DAG).

A topological order of G is an ordering of the vertices in V such that, for any edge (u, v), it must hold that u precedes v in the ordering.

Example



The following are two possible topological orders:

- $h, a, b, c, d, g, i, f, e$.
- $a, h, b, c, d, g, i, f, e$.

An ordering that is not a topological order:

- $a, h, d, b, c, g, i, f, e$ (because of edge (c, d)).

Remarks:

- A directed cyclic graph has no topological orders (think: why?).
- Every DAG has a topological order.
  - This will be a corollary from our subsequent discussion.

The Topological Sort Problem

Let $G = (V, E)$ be a directed acyclic graph (DAG). The goal of topological sort is to produce a topological order of G.

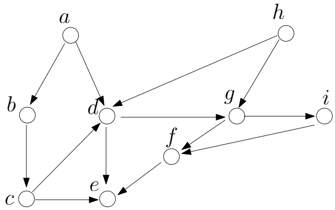
Algorithm

Very simple:

1. Create an empty list L.
2. Run DFS on G. Whenever a vertex v turns black (i.e., it is popped from the stack), append it to L.
3. Output the reverse order of L.

The total running time is clearly $O(|V| + |E|)$.

Example 1

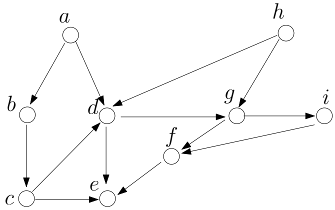


Suppose that we run DFS starting from a. The following is one possible order by which the vertices turn black:

- $e, f, i, g, d, c, b, a, h$.

Therefore, we output $h, a, b, c, d, g, i, f, e$ as a topological order.

Example 2



Suppose that we run DFS starting from d, then restarting from h, and then from a. The following is one possible order by which the vertices turn black:

- $e, f, i, g, d, h, c, b, a$.

Therefore, we output $a, b, c, h, d, g, i, f, e$ as a topological order.

We will now prove that the algorithm is correct.

Proof: Take any edge (u, v). We will show that u turns black after v, which will complete the proof.

Consider the moment when u enters the stack. We argue that currently v cannot be in the stack. Suppose that v was in the stack. As there must be a path chaining up all the vertices in the stack bottom up, we know that there is a path from v to u. Then, adding the edge (u, v) forms a cycle, contradicting the fact that G is a DAG.

Now it remains to consider:

- v is black at this moment: Then obviously u will turn black after v.
- v is white: Then by the white path theorem of DFS, we know that v will become a proper descendant of u in the DFS-forest. Therefore, u will turn back after v.

$\square$

The correctness of our algorithm also proves:

Every DAG has a topological order.