k-Selection

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In this lecture, we will put randomization to some real use, by using it to solve a non-trivial problem called k-selection elegantly and efficiently.

The k-Selection Problem

Problem: You are given a set S of n integers in an array, and also an integer $k \in [1, n]$. Design an algorithm to find the k-th smallest integer of S.

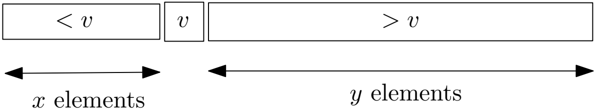
For example, suppose that $S = (53, 92, 85, 23, 35, 12, 68, 74)$, and $k = 3$. You should output 35.

This problem can be easily settled in $O(n \log n)$ time by sorting. Next, we will solve it in $O(n)$ expected time with randomization.

Idea

To illustrate the idea behind our algorithm, suppose that we pick an arbitrary element (say the first) v of S.

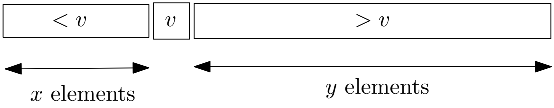
Move elements around so that those smaller than v are placed before v, and those larger are placed after v. This requires only $O(n)$ time (no sorting required).



- If $x = k - 1$, done—v is what we are looking for.
- If $x < k - 1$, recurse by performing $(k - (x + 1))$-selection on the y elements to the right of v.
- If $x > k - 1$, recurse by performing k-selection on the x elements to the left of v.

Idea

Obstacle: x or y can be very small (0 if we are unlucky) such that we can throw away only few elements before recursion!



Wish: Make $x \geq n/3$ and $y \geq n/3$.
Anecdote: Randomly select v from the whole array! Wish comes true with probability $1/3$!

New obstacle: Would still fail with probability $2/3$.
New anecdote: Choose another v if we fail—3 repeats in expectation!

Algorithm

The **rank** of an integer v in S is the number of elements in S smaller than or equal to v.

For example, suppose that $S = (53, 92, 85, 23, 35, 12, 68, 74)$. Then, the rank of 53 is 4, and that of 12 is 1.

Finding the rank of v in S (stored in an array) takes only $O(|S|)$ time.

Algorithm

1. Randomly pick an integer v from S.
2. Get the rank of v—let it be r.
3. If r is not in $[n/3, 2n/3]$, repeat from Step 1.
4. Otherwise:
   4.1 If $k = r$, return v.
   4.2 If $k < r$, produce an array A containing all the integers of S strictly smaller than v. Recurse on A by looking for the k-th smallest element in A.
   4.3 If $k > r$, produce an array A containing all the integers of S strictly larger than v. Recurse on A by looking for the $(k - r)$-th smallest element in A.

Example

Consider that we want to find the 10th smallest element from a set S of 12 elements:

17	26	38	28	41	72	83	88	5	9	12	35

Suppose that the v we randomly choose is 12, whose rank is 3. This is not in the range of $[4, 8]$

So we repeat by randomly choosing another v from S. Suppose that this time $v = 83$, whose rank is 11. This is not good either.

Repeat by choosing yet another v, say, 35, whose rank is 7. We generate an array with only the elements larger than 35:

38	41	72	83	88

Recurse by finding the 3rd smallest element in this array.

Cost Analysis

Step 1 (on Slide 7) takes $O(1)$ time.
Step 2 takes $O(n)$ time.

How many times do we have to repeat the above two steps?
With a probability $1/3$, we can proceed to Step 3 $\Rightarrow$ need to repeat only 3 times in expectation!

When we are at Step 3, A has at most $\lceil 2n/3 \rceil$ elements left.

Cost Analysis

Let $f(n)$ be the expected running time of our algorithm on an array of size n.

We know from the earlier analysis:

$$\begin{aligned} f(1) &\le O(1) \\ f(n) &\le O(n) + f(\lceil 2n/3 \rceil). \end{aligned}$$

Solving the recurrence gives $f(n) = O(n)$ (master theorem).

It is worth mentioning that the k-selection problem can actually be solved in $O(n)$ time deterministically. However, the algorithm is much more complicated—this demonstrates again the power of randomization.