Hashing

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In this lecture, we will revisit the dictionary search problem, where we want to locate an integer v in a set of size n or declare the absence of v. Recall that binary search solves the problem in $O(\log n)$ time. We will bring down the cost to $O(1)$ in expectation.

Towards the purpose, we will learn our first randomized data structure in this course. The structure is called the hash table.

The Dictionary Search Problem (Redefined)

S is a set of n integers. We want to preprocess S into a data structure so that queries of the following form can be answered efficiently:

- Given a value v, a query asks whether $v \in S$.

We will measure the performance of the data structure by examining its:

- Space consumption: How many memory cells occupied.
- Query cost: Time of answering a query.
- Preprocessing cost: Time of building the data structure.

Dictionary Search—Solution Based on Binary Search

We can solve the problem by sorting S into an array of length n, and using binary search to answer a query. This achieves:

- Space consumption: $O(n)$.
- Query cost: $O(\log n)$.
- Preprocessing cost: $O(n \log n)$.

Dictionary Search—This Lecture (the Hash Table)

We will improve the previous solution in expectation:

- Space consumption: $O(n)$.
- Query cost: $O(\log n) \Rightarrow O(1)$ in expectation.
- Preprocessing cost: $O(n \log n) \Rightarrow O(n)$.

Hashing

Main idea: divide S into a number m of disjoint subsets such that only one subset needs to be searched to answer any query.

Let us assume that every integer is in $[1, U]$ (we will revisit this assumption at the end).
Denote by $[m]$ the set of integers from 1 to m.

A **hash function** h is a function from $[U]$ to $[m]$. Namely, given any integer k, $h(k)$ returns an integer in $[m]$.

The value $h(k)$ is called the **hash value** of k.

Hash Table – Preprocessing

First, choose an integer $m > 0$, and a hash function h from $\mathbb{Z}$ to $[m]$.

Then, preprocess the input S as follows:

1. Create an array H of length m.
2. For each $i \in [1, m]$, create an empty linked list L_i. Keep the head and tail pointers of L_i in $H[i]$.
3. For each integer $x \in S$:
   - Calculate the hash value $h(x)$.
   - Insert x into $L_{h(x)}$.

Space consumption: $O(n + m)$.
Preprocessing time: $O(n + m)$.

We will always choose $m = O(n)$, so $O(n + m) = O(n)$.

Hash Table – Querying

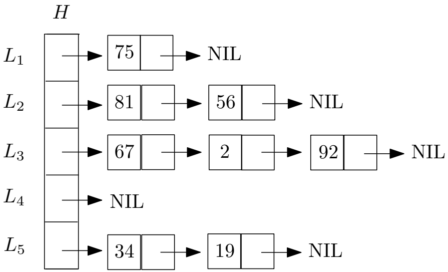
We answer a query with value v as follows:

1. Calculate the hash value $h(v)$.
2. Scan the whole $L_{h(v)}$. If v is not found, answer "no"; otherwise, answer "yes".

Query time: $O(|L_{h(v)}|)$, where $|L_{h(v)}|$ is the number of elements in $L_{h(v)}$.

Example

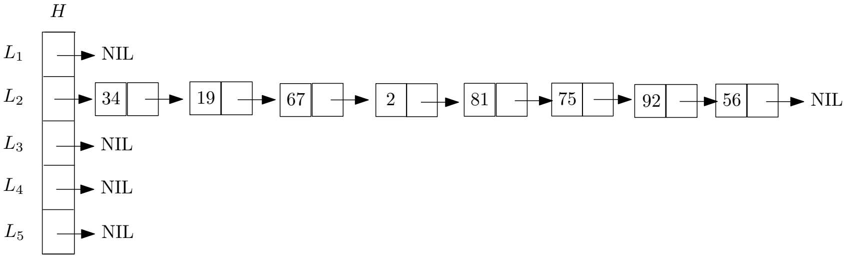
Let $S = \{34, 19, 67, 2, 81, 75, 92, 56\}$. Suppose that we choose $m = 5$, and $h(k) = 1 + (k \mod m)$.



To answer a query with search value 68, we scan all the elements in L_3, and answer "no". For this hash function, the maximum query time is the cost of scanning a linked list of 3 elements.

Example

Let $S = \{34, 19, 67, 2, 81, 75, 92, 56\}$. Suppose that we choose $m = 5$, and $h(k) = 2$.



For this hash function, the maximum query time is the cost of scanning a linked list of 8 elements (i.e., the worst possible).

It is clear that a good hash function should create linked lists of roughly the same size, i.e., "spreading out" the elements of S as evenly as possible.

Next we will introduce a technique that can choose a good hash function to guarantee $O(1)$ expected query time.

Let $\mathcal{H}$ be a family of hash functions from $[U]$ to $[m]$. $\mathcal{H}$ is universal if the following holds:

Let k_1, k_2 be two distinct integers in $[U]$. By picking a function $h \in \mathcal{H}$ uniformly at random, we guarantee that

$$\mathbf{Pr}[h(k_1) = h(k_2)] \quad \leq \quad 1/m.$$

Next, we will first prove that universality gives us the desired $O(1)$ expected query time. Then, we will describe a way to obtain such a good hash function.

Analysis of Query Time under Universality

We focus on the case where q does not exist in S (the case where it does is similar). Recall that our algorithm probes all the elements in the linked list $L_{h(q)}$. The query cost is therefore $O(|L_{h(q)}|)$.

Define random variable X_i ($i \in [1, n]$) to be 1 if the i-th element e of S has the same hash value as q (i.e., $h(e) = h(q)$), and 0 otherwise. Thus:

$$|L_{h(q)}| = \sum_{i=1}^{n} X_i$$

Analysis of Query Time under Universality

By universality, $\mathbf{Pr}[X_i = 1] \leq 1/m$, meaning that

$$\begin{aligned} \mathbf{E}[X_i] &= 1 \cdot \mathbf{Pr}[X_i = 1] + 0 \cdot \mathbf{Pr}[X_i = 0] \\ &\leq 1/m. \end{aligned}$$

Hence:

$$\mathbf{E}[|L_{h(q)}|] = \sum_{i=1}^{n} \mathbf{E}[X_i] \leq n/m.$$

By choosing $m = \Theta(n)$, we have $n/m = \Theta(1)$.

Designing a Universal Function

We now construct a universal family $\mathcal{H}$ of hash functions from $[U]$ to $[m]$.

- Pick a prime number p such that $p \geq m$ and $p \geq U$.
- For every $\alpha \in \{1, 2, ..., p-1\}$, and every $\beta \in \{0, 1, ..., p-1\}$, define:

$$h_{\alpha,\beta}(k) = 1 + (((\alpha k + \beta) \mod p) \mod m).$$

- This defines $p(p-1)$ hash functions, which constitute our $\mathcal{H}$.

The proof of universality can be found in the appendix, but will not be tested in quizzes and exams.

Existence of the Prime Number

You may be wondering why it is always possible to choose a desired prime number p.

Recall that the RAM model is defined with a word length w, namely, the number of bits in a word. Hence, $U \leq 2^w - 1$.

Number theory shows that there is at least one prime number between x and $2x$. Hence, one can prepare in advance such a prime number p in the range $[2^w, 2^{w+1}]$, and use this p to construct a universal hash family.

Remark: If n is the size of the underlying problem, the RAM model (typically) assumes that $w = \Theta(\log n)$, i.e., asymptotically the same number of bits to encode the value of n in binary.

Now we have shown that, for any set S of n integers, it is always possible to construct a hash table with the following guarantees on the dictionary search problem:

- Space $O(n)$.
- Preprocessing time $O(n)$.
- Query time $O(1)$ in expectation.

Appendix: Proof of Universality

(Will Not Be Tested)

The Prime Ring

Denote by $\mathbb{Z}_p$ the set of integers $\{0, 1, ..., p-1\}$. $\mathbb{Z}_p$ forms a commutative ring under "+" and "·", both modulo p. This means:

- $\mathbb{Z}_p$ is closed under $+$ and $\cdot$, both modulo p.
- $+$ modulo p satisfies commutativity and associativity.
  - $a + b = b + a \pmod p$ and $a + b + c = a + (b + c) \pmod p$
- $+$ modulo p has a zero element, that is, $0 + a = a \pmod p$.
- Every element a has an additive inverse $-a$, that is, $a + (-a) = 0 \pmod p$.
- $\cdot$ modulo p satisfies commutativity and associativity.
  - $a \cdot b = b \cdot a \pmod p$ and $a \cdot b \cdot c = a \cdot (b \cdot c) \pmod p$
- $\cdot$ modulo p has a one element, that is, $1 \cdot a = a \pmod a$.
- $+$ and $\cdot$ modulo p satisfy distributivity.
  - $a \cdot (b + c) = a \cdot b + a \cdot c \pmod p$
  - $(b + c) \cdot a = b \cdot a + c \cdot a \pmod p$

The Prime Ring

The ring $\mathbb{Z}_p$ has several crucial properties. Let us start with:

Lemma: Let a be a non-zero element in $\mathbb{Z}_p$. Then, $a \cdot j \neq a \cdot k \pmod{p}$ for any $j, k \in \mathbb{Z}_p$ with $j \neq k$.

Proof: Suppose without loss of generality $j > k$. Assume $a \cdot j = a \cdot k \pmod{p}$, then $a \cdot (j - k) = 0 \pmod{p}$. This means that $a \cdot (j - k)$ must be a multiple of p. Since p is prime, either a or $j - k$ must be a multiple of p. This is impossible because a and $j - k$ are non-zero elements in $\mathbb{Z}_p$. $\square$

The lemma implies that $a \cdot 0$, $a \cdot 1$, ..., $a \cdot (p - 1)$ must take unique values in $\{0, 1, ..., p - 1\}$.

The Prime Ring

The previous lemma immediately implies:

Corollary: Every non-zero element a has a unique multiplicative inverse a^{-1}, namely, $a \cdot a^{-1} = 1 \pmod{p}$.

In other words, $\mathbb{Z}_p$ is a division ring.

The Prime Ring

The next property then follows:

Lemma: Every equation $a \cdot x + b = c \pmod{p}$ where a, b, c are in $\mathbb{Z}_p$ and $a \neq 0$ has a unique solution in $\mathbb{Z}_p$.

Proof:

$$\begin{aligned} a \cdot x &= c - b \pmod{p} \\ \Rightarrow \quad x &= a^{-1} \cdot (c - b) \pmod{p} \end{aligned}$$

$\square$

Proof of Universality

Next, we will prove that the hash family $\mathcal{H}$ we constructed in Slide 15 is universal. As before, let k_1 and k_2 be distinct integers in $[U]$.

Fact 1: Let

$$\begin{aligned} g_{\alpha,\beta}(k_1) &= (\alpha \cdot k_1 + \beta) \mod p \\ g_{\alpha,\beta}(k_2) &= (\alpha \cdot k_2 + \beta) \mod p \end{aligned}$$

Then, $g_{\alpha,\beta}(k_1) \neq g_{\alpha,\beta}(k_2)$.

Proof: Otherwise, it must hold that

$$\begin{aligned} & \alpha \cdot k_1 + \beta = \alpha \cdot k_2 + \beta \pmod{p} \\ \Rightarrow \quad & \alpha \cdot (k_1 - k_2) = 0 \pmod{p} \end{aligned}$$

which is not possible. $\square$.

Proof of Universality

How many different choices are there for the pair $(g(k_1), g(k_2))$? The answer is at most $p(p-1)$ according to Fact 1: there are p^2 possible pairs in $\mathbb{Z}_p \times \mathbb{Z}_p$ but we need to exclude the p pairs where the two values are the same.

Recall that $\mathcal{H}$ has $p(p-1)$ functions.

Next, we will prove a one-to-one mapping between the possible choices of $(g(k_1), g(k_2))$ and the hash functions in $\mathcal{H}$.

Proof of Universality

Fact 2: Fix any two $x, y \in \mathbb{Z}_p$ such that $x \neq y$. There is a unique pair (α, β)—with $\alpha \in \{1, 2, ..., p-1\}$ and $\beta \in \{0, 1, ..., p-1\}$—that makes $g_{\alpha,\beta}(k_1) = x$ and $g_{\alpha,\beta}(k_2) = y$.

Proof: Suppose that h is determined by α, β selected as explained in Slide 15. Thus:

$$\begin{aligned} \alpha \cdot k_1 + \beta &= x \pmod p \\ \alpha \cdot k_2 + \beta &= y \pmod p \end{aligned}$$

Hence:

$$\begin{aligned} \alpha \cdot (k_1 - k_2) &= x - y \pmod p \\ \Rightarrow \quad \alpha &= (k_1 - k_2)^{-1} \cdot (x - y) \pmod p \\ \Rightarrow \quad \beta &= x - (k_1 - k_2)^{-1} \cdot (x - y) \cdot k_1 \pmod p \end{aligned}$$

Proof of Universality

Let P be the set of pairs (x, y) such that $x, y \in \mathbb{Z}_p$ and $x \neq y$.

We know that by choosing α, β randomly in their respective ranges, we are essentially picking a pair (x, y) for $(g_{\alpha,\beta}(k_1), g_{\alpha,\beta}(k_2))$ uniformly at random.

Notice that $h(k_1) = h(k_2)$ if and only if $g_{\alpha,\beta}(k_1) = g_{\alpha,\beta}(k_2) \pmod{m}$. So now the question boils down to: how many pairs (x, y) in P satisfy $x = y \pmod{m}$?

Proof of Universality

How many pairs (x, y) in P satisfy $x = y \pmod{m}$?

- For $x = 0$, y can take $m, 2m, 3m, \ldots$ – definitely no more that $\lceil p/m \rceil - 1 \leq (p-1)/m$ choices
- For $x = 1$, y can take $m+1, 2m+1, 3m+1, \ldots$ – definitely no more that $\lceil p/m \rceil - 1 \leq (p-1)/m$ choices
- ...

Hence, the number of such pairs is no more than $p(p-1)/m = |P|/m$.

Now we conclude that the probability of $h(k_1) = h(k_2)$ is at most $1/m$.