# Binary Search and Worst-Case Analysis

Yufei Tao

Department of Computer Science and Engineering
Chinese University of Hong Kong

A significant part of computer science is devoted to understanding the power of the RAM model in solving specific problems. Every time we discuss a problem in this course, we will learn something new.

Today's lecture is about the dictionary search problem. We will learn not only a fast algorithm for solving this problem, but also a method called worst-case analysis for measuring the quality of an algorithm.

## The Dictionary Search Problem

**Problem Input:**

In the memory, a set $S$ of $n$ integers have been arranged in ascending order at the memory cells from address 1 to $n$. The value of $n$ has been placed in Register 1 of the CPU. Another integer $v$ has been placed in Register 2 of the CPU.

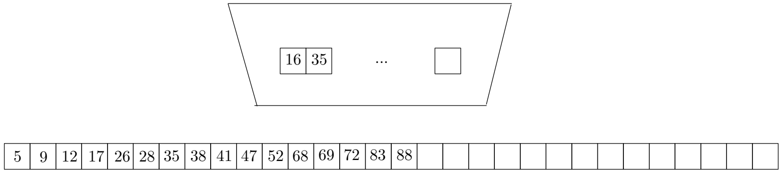
**Goal:**

Design an algorithm to determine whether $v$ exists in $S$.

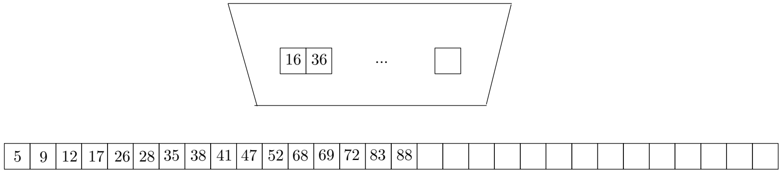
Note that we have not specified how your algorithm should indicate the outcome. This is up to you. For example, you may store 0 in a certain register to signify "no", and 1 for "yes".

We will refer to the value of $n$ as the **problem size**.

A “yes”-input with $n = 16$



A “no”-input with $n = 16$



## The First Algorithm

Let $n$ be Register 1, and $v$ be Register 2.

Simply read the memory cell of address $i$, for each $i \in [1, n]$ in turn. If any of those cells equals $v$, return yes. Otherwise, return no.

The above is a concise, but clear, description of the same algorithm as in the pseudocode of the next slide.

## The First Algorithm in Pseudocode

```
1. Let $n$ be register 1, and $v$ be register 2
2. register $i \leftarrow 1$, register $one \leftarrow 1$
3. **repeat**
4.     read into register $x$ the memory cell at address $i$
5.     **if** $x = v$ **then**
6.         **return** “yes” (by writing 1 to a register)
7.     $i \leftarrow i + one$ (effectively increasing $i$ by 1)
8. **until** $i > n$
8. **return** “no” (by writing 0 to a register)
```

## Running Time of the First Algorithm

How much time does the algorithm require? The answer depends on the problem input. Here are two extreme cases:

- If $v$ is the first element in $S$ (i.e., the integer in the memory cell of address 1), the algorithm has running time 5.
- If we are given a "no"-input, then the algorithm has running time $4n + 3$.

In computer science, it is an art to design algorithms with performance guarantees. In our scenario, this amounts to the question: what is the largest running time on the worst input with $n$ integers?

This gives rise to an important notion in the next slide.

## Worst-Case Running Time

The **worst-case cost** (or **worst-case time**) of an algorithm under a problem size $n$, is defined to be the largest running time of the algorithm on all the (possibly an infinite number of) inputs of the same size $n$.

Example

Our algorithm has worst-case time $f_1(n) = 4n + 3$.

In other words, no matter how you design the input set $S$ of $n$ integers, the algorithm always terminates with a cost at most $4n + 3$. This is its performance guarantee on every $n$.

Next, we will see another algorithm with much better worst-case time, namely, the binary search algorithm.

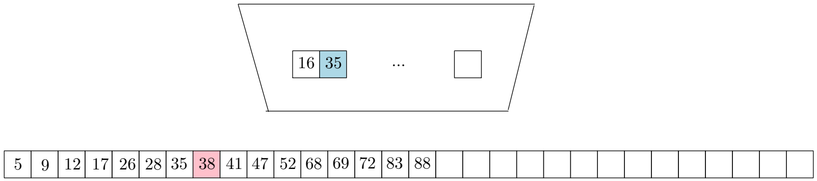
## Binary Search

We will utilize the fact that $S$ has been stored in ascending order. Let us compare $v$ to the element $x$ in the middle of $S$ (i.e., the $(n/2)$-th).

- If $v = x$, we have found $v$, and thus, can terminate.
- If $v < x$, we can immediately forget about the second half of $S$.
- If $v > x$, forget about the first half.

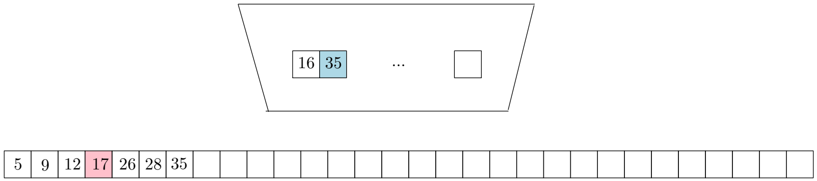
In the 2nd and 3rd cases, we have at most $n/2$ elements left.
Then, repeat the trick on those elements!

## Binary Search



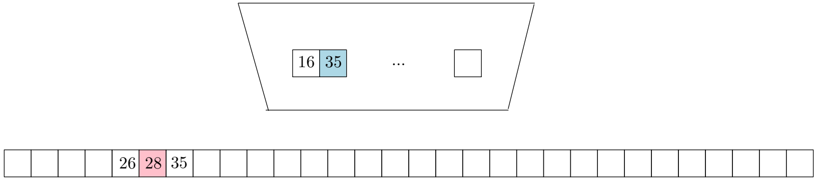
Conceptually discard the second half of $S$.

## Binary Search



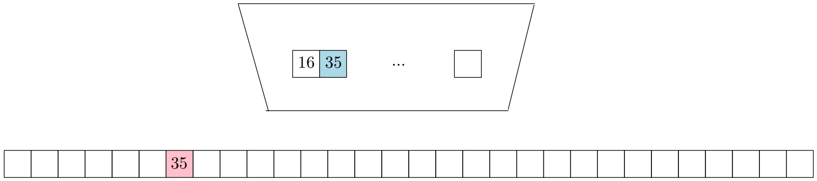
Conceptually discard the first half of what is shown.

## Binary Search



Conceptually discard the first half of what is shown.

## Binary Search



Found.

## Binary Search in Pseudocode

```
1.  let $n$ be register 1, and $v$ be register 2
2.  register $left \leftarrow 1$, $right \leftarrow n$
3.  **repeat**
4.      register $mid \leftarrow (left + right)/2$
5.      **if** the memory cell at address $mid = v$ **then**
6.          **return** "yes"
7.      **else if** the memory cell at address $mid > v$ **then**
8.          $right = mid - 1$
9.      **else**
10.         $left = mid + 1$
11. **until** $left > right$
12. **return** "no"
```

## Worst-Case Time of Binary Search

Let us call the integers whose memory addresses are from *left* to *right* as active elements.

Refer to Lines 3-10 as an iteration. Each iteration performs at most 7 atomic operations (try verifying this yourself).

## Worst-Case Time of Binary Search

How many iterations are there? After the first iteration, the number of active elements is at most $n/2$. After another, the number is at most $n/4$. In general, after $i$ iterations, the number drops to at most $n/2^i$.

Suppose that there are $h$ iterations in total. It holds that $h$ is the smallest integer satisfying (think: why?)

$$\frac{n}{2^h} < 1$$

which gives $h = 1 + \log_2 n$.

## Worst-Case Time of Binary Search (cont.)

In each iteration we perform only a constant number of operations — we will not analyze this constant precisely, except for pointing out a loose upper bound of 10.

The worst-case time of binary search is at most $f_2(n) = 10(1 + \log_2 n)$.

When $n$ is large, this running time is much lower than the time $4n + 3$ of our first algorithm.

In this lecture, we have got a taste of what computer science is like. We are seldom satisfied with just finding an algorithm that can correctly solve a problem. Instead, our goal is to design an algorithm with a strong performance guarantee, i.e., you must prove that it runs fast even in the worst case.