Homework Solutions #1

P1: 2 and 3.

P2: First, we observe that the first odd integer is 1, and its sum equals 1^2.

Next, we assume that the sum of first odd integers is n^2. We can write this as

$$\sum_{k=1}^{n} 2k - 1 = n^2 \text{ for } n \geq 0.$$

We now want to show that $\sum_{k=1}^{n+1} 2k - 1 = (n+1)^2$.

As we saw in the previous example, we note that the sum to the $(n+1)$-th integer contains the sum to the n-th integer, therefore we can write

$$\begin{aligned} \sum_{k=1}^{n+1} 2k - 1 &= \left(\sum_{k=1}^{n} 2k - 1 \right) + (2\,(n+1) - 1) \\ &= \left(\sum_{k=1}^{n} 2k - 1 \right) + (2n+1) \end{aligned}$$

We assumed that thr formula was correct for n, so we can substitue into

$$\sum_{k=1}^{n+1} 2k - 1 = n^2 + (2n+1)$$

The right-hand side can either be simplified to

$$\begin{aligned} n^2 + (2n+1) &= (n^2 + n) + (n+1) \\ &= n(n+1) + (n+1) \quad = (n+1)(n+1) = (n+1)^2. \end{aligned}$$

By now, we have proved the problem by mathematical induction.

P3: First, we observe that $\sum_{k=0}^{0}\binom{n}{k} = \binom{0}{0} = 1$ and $2^0 = 1$, so the formula is true for $n = 0$.

Next, we assume that $\sum_{k=1}^{n}\binom{n}{k} = 2^n$ is true for all $n \geq 0$.

Then, we want to show that $\sum_{k=0}^{n+1}\binom{n+1}{k} = 2^{n+1}$.

We have

$$\begin{aligned}\sum_{k=0}^{n+1}\binom{n+1}{k} &= \binom{n+1}{0} + \sum_{k=1}^{n+1}\binom{n+1}{k} \\ &= \binom{n+1}{0} + \sum_{k=1}^{n}\binom{n+1}{k} + \binom{n+1}{n+1} \\ &= 1 + \sum_{k=1}^{n}\binom{n+1}{k} + 1\end{aligned}$$

Now, since $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$, we can separate the middle sum and make the equation be

$$\begin{aligned}\sum_{k=0}^{n+1}\binom{n+1}{k} &= 1 + \sum_{k=1}^{n}\binom{n+1}{k} + 1 \\ &= 1 + \left\{\sum_{k=1}^{n}\left[\binom{n}{k} + \binom{n}{k-1}\right]\right\} + 1 \\ &= \left[1 + \sum_{k=1}^{n}\binom{n}{k}\right] + \left[\sum_{k=1}^{n}\binom{n}{k-1} + 1\right] \\ &= \left[\binom{n}{0} + \sum_{k=1}^{n}\binom{n}{k}\right] + \left[\sum_{k=0}^{n-1}\binom{n}{k} + \binom{n}{n}\right] \\ &= \sum_{k=0}^{n}\binom{n}{k} + \sum_{k=0}^{n}\binom{n}{k} \\ &= .2 \cdot \sum_{k=0}^{n}\binom{n}{k}\end{aligned}$$

By assumption, $\sum_{k=1}^{n}\binom{n}{k} = 2^n$, so that $2{\cdot}2^n = 2^{n+1}$. Here we prove our formula by induction.

P4:

1. $f(n) = \Theta(g(n))$
2. $f(n) = \Theta(g(n))$
3. $f(n) = \omega(g(n))$
4. $f(n) = o(g(n))$

P5: Time Complexity: $O(n^2)$. The time complexity looks more because of 3 nested loops. It can be observed that k is initialized only once in the outermost loop. The innermost loop executes at most $O(n)$ time for every iteration of the outermost loop, because k starts from $i + 2$ and goes up to n for all values of j. Therefore, the time complexity is $O(n^2)$.

Auxiliary Space: $O(1)$, No extra space is required. So space complexity is constant.