Introduction to Game Theory: Static Games

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Outline

Introduction

An interactive decision problem

- Involves two or more players
- Each make a decision in which the payoff depends on every players
- Broadly speaking, two types of game:
  - Zero-sum game: have winners and losers
  - non-zero-sum game: Can be all winners, or all losers, or both.

Definition

A static game is one in which a single decision is made by each player, and each player has no knowledge of the decision made by the other players before making their own decision. Decisions are made simultaneously (or order is irrelevant).

Prisoners' Dilemma

- Two crooks caught by the policeman.

	Prisoner 2 **(not confess)**	Prisoner 2 **(confess)**
Prisoner 1 **(not confess)**	-2, -2	-5, 0
Prisoner 1 **(confess)**	0, -5	-4, -4

- What should each prisoner do?
- Consider prisoner 1, what is the proper response?
- Consider prisoner 2, what is the proper response?
- The outcome is *both will confess*.
- This outcome is not socially efficient (from the prisoners' perspective).

Definition

A solution is Pareto optimal if no player's payoff can be increased without decreasing the payoff to another player. Such solution are also termed *socially efficient*.

Standardized Prisoner's Dilemma

Any game of the form

	Prisoner 2 **(silence)**	Prisoner 2 **(defection)**
Prisoner 1 **(silence)**	r,r	s, t
Prisoner 1 **(defection)**	t, s	p, p

with $t > r > p > s$ is called a Prisoner's Dilemma where t(temptation), r(reward), p(punishment), s(sucker)., e.g., $t = 5, r = 3, p = 1$ and $s = 0$. Outcome is *socially inefficient*.

Description

- The set of players, indexed by $i \in \{1, 2, \ldots, \}$;
- A pure strategy $\boldsymbol{S}_i$ for player i;
- Payoff for each player for every possible combination of pure strategies used by all players. Payoff of player i is (assuming two players only):

$$\pi_i(s_1, s_2) \quad \forall s_i \in \boldsymbol{S}_i.$$

- Or we can use the following notation:

$$\pi_i(s_i, s_{-i}) \quad \forall s_i \in \boldsymbol{S}_i.$$

Definition

A tabular description of a game, using pure strategies, is called the normal form or strategic form of a game.

Remark: For a static game, there is no real distinction between pure strategies and actions. In dynamic games, the distinction will become important.

Example:

For the prisoners' dilemma, the pure strategy sets are $\boldsymbol{S}_1 = \boldsymbol{S}_2 = \{\bar{C}, C\}$ and the payoffs are $\pi_1(\bar{C}, \bar{C}) = -2$, $\pi_1(\bar{C}, C) = -5$, $\pi_2(\bar{C}, C) = 0$.

Definition

A mixed strategy for player i gives the probabilities that action $s \in \boldsymbol{S}_i$ will be played. A mixed strategy is denoted as σ_i and the set of all possible mixed strategies for player i is denoted by $\sum_i$.

Remark

- If a player has a set of strategies $\boldsymbol{S} = \{s_a, s_b, s_c, \ldots\}$, then a mixed strategy can be represented as a vector of probabilities:

$$\sigma = (p(s_a), p(s_b), p(s_c), \ldots).$$

- A pure strategy can be represented as: $s_b = (0, 1, 0, \ldots)$.
- Mixed strategies can be represented as linear combination of pure strategies

$$\sigma = \sum_{s \in \boldsymbol{S}} p(s)s.$$

Remark: continue

- If player 1 (player 2) chooses pure strategy s with probability $p(s)$ ($q(s)$), the payoff for mixed strategy are:

$$\pi_i(\sigma_1, \sigma_2) = \sum_{s_1 \in \boldsymbol{S}_1} \sum_{s_2 \in \boldsymbol{S}_2} p(s_1) q(s_2) \pi_i(s_1, s_2).$$

HW: Exercise 4.1.

Definition

A strategy for player 1, σ_1, is strictly dominated by $\sigma_1^{'}$ if

$$\pi_1(\sigma_1^{'}, \sigma_2) > \pi_1(\sigma_1, \sigma_2) \quad \sigma_2 \in \sum_2.$$

Definition

A strategy for player 1, σ_1, is weakly dominated by $\sigma_1^{'}$ if

$$\pi_1(\sigma_1^{'}, \sigma_2) \geq \pi_1(\sigma_1, \sigma_2) \quad \sigma_2 \in \sum_2,$$

and

$$\exists \sigma_2^{'} \quad \text{s.t.} \quad \pi_1(\sigma_1^{'}, \sigma_2^{'}) > \pi_1(\sigma_1, \sigma_2^{'}).$$

A similar definition applies for player 2.

Example 1

	P_2 (L)	P_2 (R)
P_1 (U)	$3, 3$	$2, 2$
P_1 (D)	$2, 1$	$2, 1$

Assumptions

- Players are rational.
- Common knowledge of rationality (CKR).
- For player 1, U weakly dominates D.
- For player 2, L weakly dominates R.
- Consequently, player 1 will not play D and player 2 will not play R, leaving solution (U, L).

Example 2

- Find the outcome of the following game.

	P_2 (L)	P_2 (M)	P_2 (R)
P_1 (U)	$1, 0$	$1, 2$	$0, 1$
P_1 (D)	$0, 3$	$0, 1$	$2, 0$

- Solution: (U, M).

Example 3

- The following example shows that for *weakly dominated strategies*, the solution may depend on the *order* in which strategies are eliminated.
- Find the outcome of the following game.

	P_2 (L)	P_2 (M)	P_2 (R)
P_1 (U)	$10, 0$	$5, 1$	$4, -2$
P_1 (D)	$10, 1$	$5, 0$	$1, -1$

- If player 1 goes first, the outcome is (U, M).
- If player 2 goes first, he eliminates R, there is no dominated strategy. Four possible solutions: $(U, L), (U, M), (D, L)$ and (D, M).

HW: Exercise 4.2.

Example A

- Consider the following game:

	P_2 (L)	P_2 (M)	P_2 (R)
P_1 (U)	$1, 3$	$4, 2$	$2, 2$
P_1 (C)	$4, 0$	$0, 3$	$4, 1$
P_1 (D)	$2, 5$	$3, 4$	$5, 6$

- Neither player has any dominated strategies.
- Nevertheless, there is an "obvious" solution, (D, R), which maximizes the payoff of both players.
- Is it possible to define a solution in terms of something other than the elimination of dominated strategies that both identifies such obvious solutions, and keep many of the results derived using dominance techniques? **Yes.**

Definition

A Nash equilibrium (for two player games) is a pair of strategies (σ_1^*, σ_2^*) such that

$$\pi_1(\sigma_1^*, \sigma_2^*) \geq \pi_1(\sigma_1, \sigma_2^*) \quad \forall \sigma_1 \in \sum_1$$

and

$$\pi_2(\sigma_1^*, \sigma_2^*) \geq \pi_2(\sigma_1^*, \sigma_2) \quad \forall \sigma_2 \in \sum_2.$$

Remark

In other words, given the strategy adopted by the other player, neither player could do strictly better (i.e., increase their payoff) by adopting another strategy.

Solution to Example A

Solution

- Let $\sigma_2^* = R$ and let $\sigma_1 = (p, q, 1-p-q)$. We have

$$
\begin{aligned}
\pi_1(\sigma_1, R) &= 2p + 4q + 5(1-p-q) = 5 - 3p - q \\
&\leq 5 = \pi_1(D, R).
\end{aligned}
$$

- let $\sigma_1^* = D$ and let $\sigma_2 = (p, q, 1-p-q)$. We have

$$
\begin{aligned}
\pi_2(D, \sigma_2) &= 5p + 4q + 6(1-p-q) = 6 - p - 2q \\
&\leq 6 = \pi_2(D, R).
\end{aligned}
$$

- Consequently, the pair (D, R) constitutes a Nash equilibrium.

HW: Exercise 4.3.

Comments

- Nash equilibrium never includes strictly dominated strategies.
- It may include weakly dominated strategies.

Nash Equilibrium

- What we had is a procedure to "*check*" whether a point is a Nash equilibrium.
- It will be good to have an alternative method (or definition) to find the Nash equilibrium.

Definition

A strategy for player 1, $\hat{\sigma}_1$, is a best response to some (fixed) strategy for player 2, σ_2, if

$$\hat{\sigma}_1 \in argmax_{\sigma_1 \in \sum_1} \pi_1(\sigma_1, \sigma_2).$$

Similarly, $\hat{\sigma}_2$, is a best response to some σ_1 if

$$\hat{\sigma}_2 \in argmax_{\sigma_2 \in \sum_2} \pi_1(\sigma_1, \sigma_2).$$

Definition

A pair of strategies (σ_1^*, σ_2^*) is a Nash equilibrium if

$$\sigma_1^* \in argmax_{\sigma_1 \in \sum_1} \pi_1(\sigma_1, \sigma_2^*).$$

and

$$\sigma_2^* \in argmax_{\sigma_2 \in \sum_1} \pi_1(\sigma_1^*, \sigma_2).$$

How to find Nash Equilibrium

Matching Pennies

- Two players each play a penny on a table. Either "heads up" or "tails up". If the pennies match, player 1 wins the pennies; if the pennies differ, then player 2 wins the pennies.
- Game representation

	P_2 (H)	P_2 (T)
P_1 (H)	$1, -1$	$-1, 1$
P_1 (T)	$-1, 1$	$1, -1$

- Is there any pure strategy pair that is a Nash equilibrium? (go through the *loop* !).

Analysis

- Let $\sigma_1 = (p, 1-p)$ and $\sigma_2 = (q, 1-q)$.
- Payoff of player 1 is

$$\begin{aligned}
\pi_1(\sigma_1, \sigma_2) &= pq - p(1-q) - (1-p)q + (1-p)(1-q) \\
&= 1 - 2q + 2p(2q-1)
\end{aligned}$$

- Clearly, if $q < 1/2$, player 1's best response is $p = 0$ (i.e., $\hat{\sigma}_1 = (0, 1)$, "play Tails").
- If $q > 1/2$, player 1's best response is $p = 1$ (i.e., $\hat{\sigma}_1 = (1, 0)$, "play Heads").
- If $q = 1/2$, then every mixed (and pure) strategy is a best response.

Analysis: continue

- Consider player 2's payoff:

$$\begin{aligned}\pi_2(\sigma_1, \sigma_2) &= -pq + p(1-q) + (1-p)q - (1-p)(1-q) \\ &= -1 + 2p + 2q(1-2p)\end{aligned}$$

- Clearly, if $p < 1/2$, player 2's best response is $q = 1$ (i.e., $\hat{\sigma}_2 = (1, 0)$, "play Heads").
- If $p > 1/2$, player 2's best response is $q = 0$ (i.e., $\hat{\sigma}_2 = (0, 1)$, "play Tails").
- If $p = 1/2$, then every mixed (and pure) strategy is a best response.

Solution

So the only pair of strategies for which each is the best response: $\sigma_1^* = \sigma_2^* = (1/2, 1/2)$. The expected payoffs for each player are $\pi_i(\sigma_1^*, \sigma_2^*) = 0$.

Homework: Exercise 4.4.

Theorem

Suppose there exists a pair of pure strategies (s_1^*, s_2^*) such that

$$
\begin{aligned}
\pi_1(s_1^*, s_2^*) &\geq \pi_1(s_1, s_2^*) \quad \forall s_1 \in \boldsymbol{S}_1, \text{ and} \\
\pi_2(s_1^*, s_2^*) &\geq \pi_1(s_1^*, s_2) \quad \forall s_2 \in \boldsymbol{S}_2,
\end{aligned}
$$

then (s_1^*, s_2^*) is a Nash equilibrium.

Proof

- For all $\sigma_1 \in \sum_1$ we have

$$\begin{aligned} \pi_1(\sigma_1, s_2^*) &= \sum_{s \in \boldsymbol{S}_1} p(s)\pi_1(s_1, s_2^*) \\ &\leq \sum_{s \in \boldsymbol{S}_1} p(s)\pi_1(s_1^*, s_2^*) = \pi_1(s_1^*, s_2^*). \end{aligned}$$

- For all $\sigma_2 \in \sum_2$ we have

$$\begin{aligned} \pi_2(\sigma_1^*, s_2) &= \sum_{s \in \boldsymbol{S}_2} q(s)\pi_2(s_1^*, s_2) \\ &\leq \sum_{s \in \boldsymbol{S}_2} q(s)\pi_2(s_1^*, s_2^*) = \pi_2(s_1^*, s_2^*). \end{aligned}$$

- Hence, (s_1^*, s_2^*) is a Nash equilibrium.

Example

- Consider the following game:

	P_2 (L)	P_2 (M)	P_2 (R)
P_1 (U)	$1, \underline{3}$	$\underline{4}, 2$	$2, 2$
P_1 (C)	$\underline{4}, 0$	$0, \underline{3}$	$4, 1$
P_1 (D)	$2, 5$	$3, 4$	$\underline{5}, \underline{6}$

- Payoffs corresponding to a pure strategy that is a best response to one of the opponent's pure strategies are underlined.
- Two underlinings coincide for entry $(5, 6)$ or entry (D, R).
- So, D is the best response to R and vice versa.
- (D, R) is the Nash equilibrium.

HW: Exercise 4.5.

Exercise

A man has two sons. When he dies, the value of his estate is \$1000. In his will it states the two sons must each specify an amount s_i that they are willing to accept. If $s_1 + s_2 \leq 1000$, then each gets the money he asked for and the remainder goes to a church. If $s_1 + s_2 > 1000$, then neither son receives any money and \$1000 goes to a church. Assume (a) the two men care only the amount they will get; (b) they can only ask in unit of a dollar. Find **all the pure strategy Nash equilibria** of the game.

In the game of *matching pennies*, we discovered that any strategy is a best response to the Nash equilibrium strategy of the other players. Let us show this in general. To begin we, we need the following definition.

Definition

The support of a strategy σ is the set $\boldsymbol{S}(\sigma) \subseteq \boldsymbol{S}$ for all the strategies for which σ specifies $p(s) > 0$.

Example

Suppose an individual's pure strategy set is $\boldsymbol{S} = \{L, M, R\}$. Consider a mixed strategy of the form $\sigma = (p, 1-p, 0)$ and $0 < p < 1$.
$\boldsymbol{S}(\sigma) = \{L, M\}$.

Theorem: Equality of Payoffs

Let (σ_1^*, σ_2^*) be a Nash equilibrium, and let $\boldsymbol{S}_1^*$ be the support of σ_1^*. Then $\pi_1(s, \sigma_2^*) = \pi_1(\sigma_1^*, \sigma_2^*)$, $\forall s \in \boldsymbol{S}_1^*$.

Proof

If $\boldsymbol{S}_1^*$ contains one strategy, then it is trivial. When $\boldsymbol{S}_1^*$ contains more than one strategy, if the theorem is not true, then at least one strategy gives higher payoff to player 1 than $\pi_1(\sigma_1^*, \sigma_2^*)$. Let s' be that strategy, then

$$\begin{aligned}
\pi_1(\sigma_1^*, \sigma_2^*) &= \sum_{s \in \boldsymbol{S}_1^*} p^*(s)\pi_1(s, \sigma_2^*) \\
&= \sum_{s \neq s'} p^*(s)\pi_1(s, \sigma_2^*) + p^*(s')\pi_1(s', \sigma_2^*) \\
&< \sum_{s \neq s'} p^*(s)\pi_1(s', \sigma_2^*) + p^*(s')\pi_1(s', \sigma_2^*) = \pi_1(s', \sigma_2^*).
\end{aligned}$$

This contradicts the original assumption that (σ_1^*, σ_2^*) is a Nash equilibrium.

Comment

- Since all strategies $s \in \boldsymbol{S}_1^*$ give the same payoff as the randomized strategy σ_1^*, why does player 1 need to randomize?
- If player 1 were to deviate from σ_1^*, then σ_2^* would no longer be a best response and the equilibrium would disintegrate.

Example

- Consider the "Matching Pennies" Game. Suppose player 2 plays H with probability q and T with $1-q$.
- If player 1 is playing a mixed strategy at the NE, then

$$
\begin{aligned}
\pi_1(H,\sigma_2^*) &= \pi_1(T,\sigma_2^*) \\
\Leftrightarrow q\pi_1(H,H)+(1-q)\pi_1(H,T) &= q\pi_1(T,H)+(1-q)\pi_1(T,T) \\
\Leftrightarrow q-(1-q) &= -q+(1-q) \\
\Leftrightarrow 4q &= 2 \\
\Leftrightarrow q &= \frac{1}{2}
\end{aligned}
$$

- The same argument applies with the players swapped over. So the NE is $(\sigma_1^*,\sigma_2^*)=((1/2,1/2),(1/2,1/2))$.

HW: Exercise 4.7.

Nash's Theorem

Every game that has a finite strategic form (i.e., with finite number of players and finite number of pure strategies for each player) has **at least** one Nash equilibrium (involving pure or mixed strategies).

Remark

For formal proof, please refer to the book by Fudenberg and Tirole.

Existence of NE for a simple case

Every two player, two action game has *at least* one Nash equilibrium.

Proof

- Consider the following game

	P_2 (L)	P_2 (R)
P_1 (U)	a, b	c, d
P_1 (D)	e, f	g, h

- First, consider pure-strategy NE: if $a \geq e$ and $b \geq d$, then (U, L) is a NE; if $e \geq a$ and $f \geq h$, then (D, L) is a NE; if $c \geq g$ and $d \geq b$, then (U, R) is a NE; if $g \geq c$ and $h \geq f$, then (D, R) is a NE.
- There is no pure strategy NE if:
  - $a < e$, $f < h$ and $g < c$ and $d < b$, or
  - $a > e$, $f > h$ and $g > c$ and $d > b$.

Proof: continue with mixed strategy

- Using the "Equality of Payoff" Theorem. let $\sigma_1^* = (p^*, 1 - p^*)$ and $\sigma_2^* = (q^*, 1 - q^*)$, then

$$\begin{aligned} \pi_1(U, \sigma_2^*) &= \pi_1(D, \sigma_2^*) \\ \Leftrightarrow aq^* + c(1 - q^*) &= eq^* + g(1 - q^*) \\ \Leftrightarrow q^* &= \frac{(c - g)}{(c - g) + (e - a)} \end{aligned}$$

- Similarly

$$\begin{aligned} \pi_2(\sigma_1^*, L) &= \pi_2(\sigma_1^*, R) \\ \Leftrightarrow bp^* + f(1 - p^*) &= dp^* + h(1 - p^*) \\ \Leftrightarrow p^* &= \frac{(h - f)}{(h - f) + (b - d)} \end{aligned}$$

- In both cases, we require $0 < p^*, q^* < 1$ for a mixed strategy NE.

Homework

- Consider the following game with 2 players:

	P_2 (A)	P_2 (B)
P_1 (A)	a, a	b, c
P_1 (B)	c, b	d, d

- Show that such a game has at least one symmetric Nash Equilibrium.

Example

- The following game is "Battle of the Sexes": Husband prefers to watch the football (F), wife prefers to watch the soap opera (S).

	Wife (F)	Wife (S)
Husband (F)	$3, 2$	$1, 1$
Husband (S)	$0, 0$	$2, 3$

- Using the "best response method", there are two pure-strategy Nash equilibria: $(F, F), (S, S)$.
- Using the "Equality of Payoffs Theorem", we can find a mixed strategy at Nash equilibrium (σ_h^*, σ_w^*) with $\sigma_h^* = (p(F), p(S)) = (3/4, 1/4)$ and $\sigma_w^* = (q(F), q(S)) = (1/4, 3/4)$.
- Although we have three NEs, how should player decide?
- For the randomizing NE, the asymmetric outcomes can occur (e.g., (F,S) or (S,F)). The most likely outcome is (F, S) which occurs with probability $9/16$.

Definition

A *generalized affine transformation* of the payoffs for player 1 is

$$\pi_1^{'}(s_1, s_2) = \alpha_1 \pi_1(s_1, s_2) + \beta_1(s_2) \quad \forall s_1 \in \boldsymbol{S}_1, \alpha_1 > 0, \beta_1(s_2) \in \mathbf{R}.$$

Similarly, an affine transformation of the payoffs for player 2 is

$$\pi_2^{'}(s_1, s_2) = \alpha_2 \pi_2(s_1, s_2) + \beta_2(s_1) \quad \forall s_2 \in \boldsymbol{S}_2, \alpha_2 > 0, \beta_2(s_1) \in \mathbf{R}.$$

Example

	$P_2(L)$	$P_2(R)$
P_1 (U)	$3, 3$	$0, 0$
P_1 (D)	$-1, 2$	$2, 8$

$\Longrightarrow$

	$P_2(L)$	$P_2(R)$
P_1 (U)	$2, 1$	$0, 0$
P_1 (D)	$0, 0$	$1, 2$

where $\alpha_1 = 1/2$, $\beta_1(L) = 1/2$, $\beta_1(R) = 0$, $\alpha_2 = 1/3$, $\beta_2(U) = 0$, $\beta_2(D) = -2/3$. What are their NEs? Do they use the same strategies?

Theorem

If the payoff table is altered by generalized affine transformations, the set of Nash equilibria is unaffected (although the payoffs at those equilibria do change).

Proof

$$
\begin{aligned}
\pi_1^{'}(\sigma_1^*, \sigma_2^*) &\geq \pi_1^{'}(\sigma_1, \sigma_2^*), \\
\sum_{s_1}\sum_{s_2} p^*(s_1)q^*(s_2)\pi_1^{'}(s_1, s_2) &\geq \sum_{s_1}\sum_{s_2} p(s_1)q^*(s_2)\pi_1^{'}(s_1, s_2),
\end{aligned}
$$

$$
\alpha_1 \sum_{s_1}\sum_{s_2} p^*(s_1)q^*(s_2)\pi_1(s_1, s_2) + \sum_{s_1}\sum_{s_2} p^*(s_1)q^*(s_2)\beta_2(s_2)
$$

$$
\geq \alpha_1 \sum_{s_1}\sum_{s_2} p(s_1)q^*(s_2)\pi_1(s_1, s_2) + \sum_{s_1}\sum_{s_2} p(s_1)q^*(s_2)\beta_2(s_2),
$$

Proof: continue

$$\alpha_1 \sum_{s_1} \sum_{s_2} p^*(s_1) q^*(s_2) \pi_1(s_1, s_2) + \sum_{s_2} q^*(s_2) \beta_2(s_2)$$
$$\geq \alpha_1 \sum_{s_1} \sum_{s_2} p(s_1) q^*(s_2) \pi_1(s_1, s_2) + \sum_{s_2} q^*(s_2) \beta_2(s_2),$$

$$\begin{aligned}
\alpha_1 \sum_{s_1} \sum_{s_2} p^*(s_1) q^*(s_2) \pi_1(s_1, s_2) &\geq \alpha_1 \sum_{s_1} \sum_{s_2} p(s_1) q^*(s_2) \pi_1(s_1, s_2) \\
\sum_{s_1} \sum_{s_2} p^*(s_1) q^*(s_2) \pi_1(s_1, s_2) &\geq \sum_{s_1} \sum_{s_2} p(s_1) q^*(s_2) \pi_1(s_1, s_2) \\
\pi_1(\sigma_1^*, \sigma_2^*) &\geq \pi_1(\sigma_1, \sigma_2^*)
\end{aligned}$$

The analogous argument for player 2 completes the proof.

Generic and Non-generic Games

Definition

A *generic game* is one in which a small change (or non-affine transformation) of any *one* of the payoffs does not introduce new Nash equilibria or remove existing ones. In practice, this means that there should be no equalities between payoffs that are compared to determine a Nash equilibrium.

Example of non-generic game

- Games we have examined (e.g., prisoners dilemma, matching pennies, battles of sexes) have been generic.
- Consider the following non-generic game:

	P_2 (L)	P_2 (M)	P_2 (R)
P_1 (U)	$10, 0$	$5, 1$	$4, -2$
P_1 (D)	$10, 1$	$5, 0$	$1, -1$

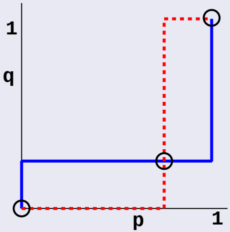
- It is non-generic
- (D, L) is a NE, but player 1 gets the same payoff by playing U rather than D (against L).
- Similarly, (U, M) is a NE, but player 1 gets the same payoff by playing D rather than U (against M).

Oddness Theorem

All generic games have an "odd" number of Nash equilibria.

Remark

- A formal proof is rather difficult. Reading assignment.
- Consider the following illustration of the "Battle of Sexes":



- In contrast, the number of Nash equilibria in a non-generic game is (usually) infinite !!!

- Consider the previous "non-generic" game. Let $\sigma_1 = (p, 1-p)$ and $\sigma_2 = (q, r, 1-q-r)$, then

$$\begin{aligned}\pi_1(\sigma_1, \sigma_2) &= 1 + 9q + 4r + 3p(1-q-r) \\ \pi_2(\sigma_1, \sigma_2) &= -(1+p) + 2q + r(1+2p).\end{aligned}$$

- The best responses are

$$\hat{\sigma}_1 = \begin{cases} (1,0) & \text{if } q+r<1, \\ (x, 1-x) & \text{if } q+r=1, \text{ with } x \in [0,1]. \end{cases}$$

$$\hat{\sigma}_2 = \begin{cases} (1,0,0) & \text{if } p<1/2, \\ (0,1,0) & \text{if } p>1/2, \\ (y, 1-y, 0) & \text{if } p=1/2, \text{ with } y \in [0,1]. \end{cases}$$

- NEs: (1) $\sigma_1^* = (x, 1-x)$ with $x \in [0, 1/2)$ and $\sigma_2^* = (1,0,0)$, (2) $\sigma_1^* = (x, 1-x)$ with $x \in (1/2, 1]$ and $\sigma_2^* = (0,1,0)$, (3) $\sigma_1^* = (1/2, 1/2)$ and $\sigma_2^* = (y, 1-y, 0)$ with $y \in [0,1]$.

HW: Exercise 4.10.

Zero-sum Games

- A zero-sum game is one in which the payoffs to the players add up to zero, e.g., "Matching Pennies" is a zero-sum game. If player 1 uses a strategy $\sigma_1 = (p, 1-p)$ and player 2 uses a strategy $\sigma_2 = (q, 1-q)$, we have:

$$\begin{aligned} \pi_1(\sigma_1, \sigma_2) &= pq + -p(1-q) - (1-p)q + (1-p)(1-q) \\ &= (2p-1)(2q-1) = -\pi_2(\sigma_1, \sigma_2) \end{aligned}$$

  In other words, the interests of the players are exactly *opposed*: one only wins what the other loses.
- Zero-sum games were first type of games to be studied (before J. Nash).
- Zero-sum games were solved by finding the "minimax" solution.

Claim:

- Define $\pi(\sigma_1, \sigma_2) = \pi_1(\sigma_1, \sigma_2) = -\pi_2(\sigma_1, \sigma_2)$. The NE conditions:

$$
\begin{aligned}
\pi_1(\sigma_1^*, \sigma_2^*) &\geq \pi_1(\sigma_1, \sigma_2^*) \quad \forall \sigma_1 \in \textstyle\sum_1 \\
\pi_2(\sigma_1^*, \sigma_2^*) &\geq \pi_2(\sigma_1^*, \sigma_2) \quad \forall \sigma_2 \in \textstyle\sum_2.
\end{aligned}
$$

- The above NE conditions can be rewritten:

$$
\begin{aligned}
\pi(\sigma_1^*, \sigma_2^*) &= \max_{\sigma_1 \in \sum_1} \pi(\sigma_1, \sigma_2^*) \\
\pi(\sigma_1^*, \sigma_2^*) &= \min_{\sigma_2^* \in \sum_2} \pi(\sigma_1^*, \sigma_2).
\end{aligned}
$$

Claim continue

- Since both players should play a best response to the other's strategy, these two conditions can be combined as:

$$\pi(\sigma_1^*, \sigma_2^*) = \max_{\sigma_1 \in \sum_1} \pi(\sigma_1, \sigma_2^*) = \max_{\sigma_1 \in \sum_1} \min_{\sigma_2 \in \sum_2} \pi(\sigma_1, \sigma_2).$$

- Equivalently,

$$\pi(\sigma_1^*, \sigma_2^*) = \min_{\sigma_2 \in \sum_2} \pi(\sigma_1^*, \sigma_2) = \min_{\sigma_2 \in \sum_2} \max_{\sigma_1 \in \sum_1} \pi(\sigma_1, \sigma_2).$$

- In other words, player 1 uses the "maximin" while player 2 uses the "minimax".

Application of Min-max algorithm

- Consider the following game:

	A	B	C	D	
A	4, -4	3, -3	2, -2	5, -5	2
B	-10,10	2, -2	0, 0	-1, 1	-10
C	7,-7	5, -5	1, -1	3, -3	1
D	0,0	8, -8	-4, 4	-5, 5	-5
	7	8	2	5	

- Player 1: choose minimum for each row, then find maximum entry (or the maximin operation).
- Player 2: choose maximum for each column, then find minimum entry (or the minimax operation).
- If maximin $=$ minimax, it is the Nash equilibrium.

Comment on the Min-max algorithm

- In general, zero-sum game can have multiple NE.

	A	B	C	D	
A	3, -3	2, -2	2, -2	5, -5	2
B	2,-2	-10,10	0, 0	-1, 1	-10
C	5,-5	2, -2	2, -2	3, -3	2
D	8,-8	0,0 8	-4, 4	-5, 5	-5
	8	2	2	5	

- Same payoff in every NE.
- Strategies are interchangeable. Example, strategies (A, B) and (C, C) are NE, then (A, C) and (C, B) are also NE.

Note that for zero-sum game without pure strategies NE, we have to consider mixed strategies.

Application of the Equality of Payoffs Theorem

Exercise

An "Ace-King-Queen" game with two players. Each player bets \$5. Each player chooses a card from the set $\{$Ace(A),King(K),Queen(Q)$\}$. The winning rule is: A beats K; K beats Q; Q beats A. The winning player takes the \$10 in the pot. If both players choose the same card (both A, both K, or both Q), the game is drawn and \$5 stake is returned to each player. **What is the unique Nash equilibrium for this game?**

Theorem on Zero-sum Game

A generic zero-sum game has a unique solution.

Proof

Refer to Von Neuman and Morgenstern, "*Theory of Games and Economic Behavior*", 3rd edition, Princeton University, 1953.

Games with $n-$players

- Label player by $i \in \{1, 2, \ldots, n\}$.
- Player i has a set of pure strategies $\boldsymbol{S}_i$ and the corresponding mixed strategies $\sum_i$.
- Payoff of player i depends on the list of strategies $\sigma_1, \sigma_2, \ldots, \sigma_n$.
- We also use σ_{-i} to denote the list of strategies used by all players except player i. So payoff for player i is $\pi_i(\sigma_i, \sigma_{-i})$.
- Suppose player i uses a mixed strategy σ_i which specifies playing pure strategies $s \in \boldsymbol{S}_i$ with $p_i(s)$. Payoff is

$$\pi_i(\sigma_i, \sigma_{-i}) = \sum_{s_1 \in \boldsymbol{S}_1} \cdots \sum_{s_n \in \boldsymbol{S}_n} p_1(s_1) \cdots p_n(s_n) \pi_i(s_1, \ldots, s_n).$$

- A NE in a $n-$player game is a list of mixed strategies $\sigma_1^*, \ldots, \sigma_n^*$ such that

$$\sigma_i^* \in argmax_{\sigma_i \in \sum_i} \pi_i(\sigma_i, \sigma_{-i}^*) \quad \forall i \in \{1, 2, \ldots, n\}.$$

Example

- A static three-player game: P_1 chooses U and D; P_2 chooses L and R; P_3 chooses A and B.
- Instead of representing a $3-$dimensional payoff table, we have the following payoff tables: when P_3 plays A or B:

A	$P_2(L)$	$P_2(R)$
P_1 (U)	$1, 1, 0$	$2, 2, 3$
P_1 (D)	$2, 2, 3$	$3, 3, 0$

B	$P_2(L)$	$P_2(R)$
P_1 (U)	$-1, -1, 2$	$2, 0, 2$
P_1 (D)	$0, 2, 2$	$1, 1, 2$

Continue:

- Suppose P_3 chooses A, the best responses for P_1 and P_2 are strategies $\hat{\sigma}_1 = \hat{\sigma}_2 = (0, 1)$. Note that this is **NOT** a NE because choosing A may not be the best response of P_3.
- Suppose P_3 chooses B, the best responses for P_1 and P_2 are strategies $\hat{\sigma}_1 = \hat{\sigma}_2 = (1/2, 1/2)$.
- By playing B, P_3 gains 2.
- The NE $(\sigma_1^*, \sigma_2^*, \sigma_3^*)$ with

$$\sigma_1^* = (1/2, 1/2); \quad \sigma_2^* = (1/2, 1/2); \quad \sigma_3^* = (0, 1).$$

HW: Exercise 4.14.