Questions

1. A Chicken lays a Poisson(\(\lambda\)) number \(N\) of eggs. Each egg independently hatches a chick with probability \(p\). Let \(X\) be the number of chicks that hatch. Calculate

(a) the conditional expectation \(E[X | N = n]\);
(b) the unconditional expectation \(E[X]\);
(c) the unconditional expectation \(E[NX]\);
(d) the covariance \(\text{Cov}[X, N]\).

2. You draw 10 balls at random among 15 red and 5 blue balls. Let \(X\) be the number of red balls drawn.

(a) What is the expected value of \(X\)?
(b) Write \(X = X_1 + X_2 + \cdots + X_{10}\), where \(X_i\) indicates if the \(i\)-th drawn ball is red. What is the variance of \(X_i\)?
(c) What is the covariance of \(X_i\) and \(X_j\) (\(i \neq j\))?
(d) What is the variance of \(X\)?

3. Two typing monkeys sit at special keyboards. The keyboards have only two letters, \(a\) and \(b\). Each monkey types in a random 200 letter string, independently of the other one. Let \(E\) be the event “There is a pattern of 20 consecutive letters that appears in both strings.” Show that \(P(E) < 5\%\).

4. 100 people put their hats in a box and each one pulls a random hat out.

(a) Let \(G\) be any 10-person group. What is the probability that everyone in \(G\) pulls their own hat?
(b) What is the expected number of 10-person groups in which everyone pulls their own hat?
(c) Show that the probability that 10 or more people pull their own hat is less than \(10^{-6}\).

Additional ESTR 2018 question

5. In ESTR 2018 Lecture 9 I claimed that \(E[X^4] = 3\sigma^4\) for a Normal(0, \(\sigma\)) random variable \(X\). In this exercise you will derive that formula.

(a) Let \(X\) be a Normal(0, 1) random variable. Show that \(E[e^{tX}] = e^{-t^2/2}\) for every real number \(t\).
(b) Calculate \(E[X^4]\) by taking derivatives. (You may assume that the expectation of a derivative is the same as the derivative of an expectation.)
(c) Can you calculate \(E[X^k]\) for all \(k\)?