1. Let \( A \) and \( B \) be arbitrary events. Which of the following is true? If you answer yes, prove it using the axioms of probability. If you answer no, provide a counterexample.

(a) \( P(A|B) + P(A|B^c) = 1 \).

**Solution:** No. If \( B \) is the event of a fair coin flipping heads and \( A \) is the event of the coin flipping heads or tails then \( P(A|B) = 1 \) and \( P(A|B^c) = 1 \).

(b) \( P(A \cap B|A \cup B) \leq P(A|B) \).

**Solution:** Yes, because \( P(A \cup B) \geq P(B) \) and so
\[
P(A \cap B|A \cup B) = \frac{P(A \cap B)}{P(A \cup B)} \leq \frac{P(A \cap B)}{P(B)} = P(A|B).
\]

2. \( n \) independent random numbers are sampled uniformly from the interval \([0, 1]\).

(a) If \( n = 10 \), what is the probability that exactly 4 of them are greater than 0.7?

**Solution:** Let \( N \) be the number of such random numbers greater than 0.7. Then \( N \) is a binomial random variable with \( n = 10 \) samples and success probability \( p = 3 \), so
\[
P(X = 4) = \binom{10}{4} 0.3^4 (1 - 0.3)^{10-4} \approx 0.200.
\]

(b) If \( n = 50 \), use the Central Limit Theorem to estimate the probability that their sum is between 20 and 25 (inclusive).

**Solution:** Let \( X_i \) denote the value of the \( i \)-th random number \( (i = 1, \ldots, n) \). Then \( X_1, \ldots, X_n \) are independent random variables with mean 1/2 and variance 1/12. Let \( X = X_1 + \cdots + X_n \). Then \( \text{E}[X] = 25 \) and \( \text{Var}[X] = 50/12 \). By the Central Limit Theorem, the CDF of \( X \) can be approximated by the CDF of a Normal\((25, \sqrt{50}/12)\) random variable \( \bar{N} \). Normalizing \( \bar{N} = 25 + N \cdot \sqrt{50}/12 \),
\[
P(20 \leq X \leq 25) \approx P(20 \leq \bar{N} \leq 25)
\approx P(20 \leq \bar{N} \leq 25 + N \cdot \sqrt{50}/12 \leq 25)
\approx P(-5/\sqrt{50}/12 \leq N \leq 0)
\approx P(-2.45 \leq N \leq 0)
\approx F_N(0) - F_N(-2.45)
\approx 0.5 - 0.0071
\approx 0.4929.
\]
Companies A and B produce lightbulbs. Their lifetimes are exponential random variables with mean 2 years for company A and 1 year for company B.

(a) A shop sources 3/4 of its lightbulbs from company A and the remaining 1/4 from company B. If a random lightbulb from the shop survived for 2 years, how likely is it to have been produced by company B?

Solution: Let \( X \) be the lifetime of the lightbulb, and \( A \) and \( B \) be the (complementary) events that the respective company produced it. Then \( P(X \geq t | A) = e^{-t/2} \) and \( P(X \geq t | B) = e^{-t} \).

By the total probability theorem,

\[
P(X \geq 2) = P(X \geq 2 | A) P(A) + P(X \geq 2 | B) P(B) = e^{-1} \cdot \frac{3}{4} + e^{-2} \cdot \frac{1}{4} \approx 0.310.
\]

and by Bayes’ rule

\[
P(B | X \geq 2) = \frac{P(X \geq 2 | B) P(B)}{P(X \geq 2)} = \frac{e^{-2} \cdot 1/4}{e^{-1} \cdot 3/4 + e^{-2} \cdot 1/4} \approx 0.109.
\]

(b) What is the probability that a lightbulb produced by company B outlasts one produced by company A? Assume their lifetimes are independent.

Solution: Let \( X \) and \( Y \) be their respective lifetimes. The PDF of \( X \) is \( f_X(x) = \frac{1}{2}e^{-x/2} \). By the total probability theorem,

\[
P(Y > X) = \int_0^\infty P(Y > x | X = x) f_X(x) dx = \int_0^\infty P(Y > x) f_X(x) dx = \int_0^\infty e^{-x} \cdot \frac{1}{2}e^{-x/2} dx = \frac{1}{3}.
\]

Alternative solution: If we divide each year into \( n \) equal intervals and flip independent coins of probabilities 1/2\(n \) and 1/\( n \) for the failure of each bulb in each interval, then \( X \) and \( Y \) are the times of the first failure in the limit as \( n \) goes to infinity. For a fixed \( n \), let \( A_n \) be the event that lightbulb A failed in the interval in which the first lightbulb failure occurred. Then \( A_n \) has the same probability as the event that the first coin came up heads given that at least one did, namely

\[
P(A_n) = \frac{1/2n}{1 - (1 - 1/2n)(1 - 1/n)} = \frac{1/2n}{1/2n + 1/n - (1/2n)(1/n)} = \frac{1}{3 + 1/n}.
\]

As \( n \) tends to infinity, \( P(A_n) \) tends to 1/3 so \( P(Y > X) = 1/3 \).
4. Alice takes $T$ hours to travel to Bob’s house, where $T$ is a random variable with PDF $f_T(t) = \begin{cases} 1/t^2, & \text{when } t \geq 1 \\ 0, & \text{otherwise.} \end{cases}$

(a) Find the CDF (cumulative distribution function) $F_T(t) = P(T \leq t)$.

**Solution:** $F_T(t)$ is zero when $t < 1$. When $t \geq 1$,

$$F_T(t) = \int_1^t \frac{1}{u^2} \, \text{d}u = -\frac{1}{u}\bigg|_1^t = 1 - \frac{1}{t}.$$

(b) The distance between Alice’s and Bob’s house is one mile so that Alice travels at a speed $V = 1/T$ miles per hour. What is Alice’s expected speed $E[V]$?

**Solution:** The CDF $F_V(v)$ of $V$ is zero when $v \leq 0$. If $v > 0$,

$$P(V \leq v) = P(1/T \leq v) = P(T \geq 1/v) = 1 - F_T(1/v) = \begin{cases} v, & \text{if } 0 \leq v \leq 1, \\ 1, & \text{if } v \geq 1. \end{cases}$$

This is the CDF of a Uniform(0,1) random variable, so $E[V] = 1/2$.

5. A group of 10 boys and 10 girls is randomly divided into 5 teams A, B, C, D, E with 4 children per team.

(a) What is the probability that all children in team A are of the same gender?

**Solution:** By the multiplication rule, this probability is $p = 9/19 \cdot 8/18 \cdot 7/17 \approx 0.087$.

(b) Is the probability that all teams are of mixed gender more than 50% or not? Justify your answer.

**Solution:** It is. Let $S$ be the number of same gender teams. By linearity of expectation, $E[S] = 5p \approx 0.433$. By Markov’s inequality, $P(S \geq 1) \leq E[S]$, so the complementary event $S = 0$ occurs with probability at least $1 - 0.433 > 0.5$. 