

**Practice questions**

1. Alice, Bob, and Charlie are equally likely to have been born on any three days of the year. Let  $E_{AB}$  be the event that Alice and Bob were born on the same day. Define  $E_{BC}$  and  $E_{CA}$  analogously. Which of the following statements is true:
  - (a) Any two of the three events  $E_{AB}, E_{BC}, E_{CA}$  are independent.
  - (b)  $E_{AB}, E_{BC}$ , and  $E_{CA}$  are independent.
  - (c)  $E_{AB}$  and  $E_{BC}$  are independent conditioned on  $E_{CA}$ .

**Solution:** Our sample space will consist of all triples of possible birthdays  $(a, b, c)$  where  $a$ ,  $b$ , and  $c$  are numbers between 1 and 365 (we exclude February 29 to keep things simple). We assume equally likely outcomes, so all triples occur with probability  $365^{-3}$ .

- (a) **True.** The intersection of any two events is the event that all three were born on the same day. There are 365 such outcomes, each occurring with probability  $365^{-3}$ , so

$$P(E_{AB} \cap E_{BC}) = P(E_{AB} \cap E_{CA}) = P(E_{BC} \cap E_{CA}) = 365^{-2}.$$

On the other hand, probability that any two of them were born on the same day is

$$P(E_{AB}) = P(E_{BC}) = P(E_{CA}) = 365 \cdot \frac{1}{365^2} = 365^{-1}.$$

Since  $P(E_{AB} \cap E_{BC}) = 365^{-2} = P(E_{AB}) \cdot P(E_{BC})$ , the two events  $E_{AB}$  and  $E_{BC}$  are independent, and similarly for the other two pairs.

- (b) **False.**  $E_{AB} \cap E_{BC} \cap E_{CA}$  is also the event that all three were born on the same day, so  $P(E_{AB} \cap E_{BC} \cap E_{CA}) = 365^{-3}$ . On the other hand  $P(E_{AB}) \cdot P(E_{BC}) \cdot P(E_{CA}) = 365^{-3}$  so the three events are not independent.
- (c) **False.** Conditional independence holds when

$$P(E_{BC}|E_{CA}) = P(E_{BC}|E_{CA} \cap E_{AB}).$$

The probability on the left is the ratio of the probabilities of  $E_{BC} \cap E_{CA}$  and  $E_{CA}$ , so it equals  $365^{-1}$ . The probability on the right equals one, because if Alice and Bob share a birthday and Alice and Charlie also do, so will Bob and Charlie.

2. You go to a party with 500 guests. What is the probability that exactly one other guest has the same birthday as you? Calculate this exactly and also approximately by using the Poisson PMF  $p(k) = \lambda^k e^{-\lambda} / k!$ . (For simplicity, exclude birthdays on February 29.) (*Textbook problem 2.2*)

**Solution:** We can model the number of guests having *your* birthday as a Binomial( $n = 499, p = 1/365$ ) random variable  $X$ . The probability that  $X = 1$  is  $\binom{499}{1} \cdot p \cdot (1-p)^{499-1} \approx 0.3486$ . Alternatively, we can model this process as a Poisson( $\lambda$ ) random variable  $Y$  with  $\lambda = np = 499/365$ . Then the probability of  $Y = 1$  is  $\lambda \cdot e^{-\lambda} \approx 0.3483$ .

3. Let  $X$  be the number of 1's you observed after rolling a fair die for  $n$  times. Show that PMF of  $X$  can be computed by starting with  $p_X(0) = (1 - \frac{1}{6})^n$  and then using the recursive formula

$$p_X(k+1) = \frac{1}{5} \cdot \frac{n-k}{k+1} \cdot p_X(k), \quad k = 0, 1, \dots, n-1.$$

(Adapted from Textbook problem 2.8)

**Solution:**  $X$  is a Binomial( $n, p = 1/6$ ) random variable. So  $p_X(0) = (1 - \frac{1}{6})^n$  is probability that none of the  $n$  rolls gives 1.

For any  $0 \leq k \leq n$ , PMF of Binomial RV with parameter  $n$  and  $p$  is known to be

$$p_X(k) = \binom{n}{k} \cdot p^k (1-p)^{n-k} = \frac{n!}{k!(n-k)!} \cdot p^k (1-p)^{n-k}.$$

Then for any  $0 \leq k \leq n-1$ , substitute  $k+1$  into above equation:

$$\begin{aligned} p_X(k+1) &= \binom{n}{k+1} \cdot p^{k+1} (1-p)^{n-k-1} = \frac{n!}{(k+1)!(n-k-1)!} \cdot p^{k+1} (1-p)^{n-k-1} \\ &= \frac{n!}{k!(n-k)!} \cdot \frac{n-k}{k+1} \cdot (p^k \cdot p) \cdot ((1-p)^{n-k} (1-p)^{-1}) \\ &= \frac{p}{1-p} \cdot \frac{n-k}{k+1} \cdot p_X(k) = \frac{1/6}{5/6} \cdot \frac{n-k}{k+1} \cdot p_X(k) \\ &= \frac{1}{5} \cdot \frac{n-k}{k+1} \cdot p_X(k). \end{aligned}$$

4. An ENGG 2430A tutorial meets for 11 weeks. Each week, the TA asks 5 questions and chooses 5 random but distinct students to answer them, independently of what happened in previous weeks. If you are one of 40 students in the tutorial (and attendance is always perfect!), what is the probability that you are chosen in the final week but not before that?

**Solution:** The probability that you are chosen in any given week is  $5/40$ , so the probability that you are not chosen in that week is  $35/40$ . The week in which you are first chosen is a Geometric( $5/40$ ) random variable, so the probability it takes value 11 is  $(35/40)^{10} \cdot 5/40 \approx 0.0329$ .

5. You are given a biased coin with probability  $p$  of getting head and you toss the coin for  $X$  times until you first see a head. Find the PMF of the random variable  $Y = X \bmod 3$ . For example, if the sequence is TTTH, then  $X = 4$  and  $Y = 4 \bmod 3 = 1$ .

**Solution:**  $X$  is a Geometric( $p$ ) random variable so its PMF is:

$$p_X(0) = 0, \quad p_X(k) = (1-p)^{k-1} \cdot p, \quad k = 1, 2, \dots$$

For  $k = 0$ ,  $Y = 0$  if  $X = 3, 6, 9, \dots$  and so on. Then

$$p_Y(0) = \sum_{i=1}^{\infty} p_X(3 \cdot i) = \sum_{i=1}^{\infty} (1-p)^{3 \cdot i - 1} \cdot p = p(1-p)^{-1} \sum_{i=0}^{\infty} ((1-p)^3)^i = \frac{p(1-p)^2}{1 - (1-p)^3}.$$

Similarly, for  $k = 1, 2$ ,  $Y = k$  if  $X = k, 3+k, 3+k, \dots$  and so on. Therefore

$$p_Y(1) = \sum_{i=0}^{\infty} p_X(3 \cdot i + 1) = \sum_{i=0}^{\infty} (1-p)^{3 \cdot i + 1 - 1} \cdot p = p \sum_{i=0}^{\infty} ((1-p)^3)^i = \frac{p}{1 - (1-p)^3},$$

and essentially the same calculation gives

$$p_Y(2) = \sum_{i=0}^{\infty} p_X(3 \cdot i + 2) = \frac{p(1-p)}{1 - (1-p)^3}.$$