1. A bag contains three dice with faces numbered 1, 1, 2, 2, 3, 6 and seven fair dice (with faces numbered 1, 2, 3, 4, 5, 6). A die is chosen at random from the bag and tossed. Find the probability of each outcome.

   Solution: Let Y_i denote the event that the outcome of the selected die is i, A be the event of unfair die being selected, B be the event of fair die being selected. Then,

$$\begin{aligned}
\mathrm{P}(Y_1) &= \mathrm{P}(Y_1|A)\,\mathrm{P}(A) + \mathrm{P}(Y_1|B)\,\mathrm{P}(B) = (1/3)\cdot 0.3 + (1/6)\cdot 0.7 = 13/60,\\
\mathrm{P}(Y_2) &= \mathrm{P}(Y_2|A)\,\mathrm{P}(A) + \mathrm{P}(Y_2|B)\,\mathrm{P}(B) = (1/3)\cdot 0.3 + (1/6)\cdot 0.7 = 13/60,\\
\mathrm{P}(Y_3) &= \mathrm{P}(Y_3|A)\,\mathrm{P}(A) + \mathrm{P}(Y_3|B)\,\mathrm{P}(B) = (1/6)\cdot 0.3 + (1/6)\cdot 0.7 = 1/6,\\
\mathrm{P}(Y_4) &= \mathrm{P}(Y_4|A)\,\mathrm{P}(A) + \mathrm{P}(Y_4|B)\,\mathrm{P}(B) = (0)\cdot 0.3 + (1/6)\cdot 0.7 = 7/60,\\
\mathrm{P}(Y_5) &= \mathrm{P}(Y_5|A)\,\mathrm{P}(A) + \mathrm{P}(Y_5|B)\,\mathrm{P}(B) = (0)\cdot 0.3 + (1/6)\cdot 0.7 = 7/60,\\
\mathrm{P}(Y_6) &= \mathrm{P}(Y_6|A)\,\mathrm{P}(A) + \mathrm{P}(Y_6|B)\,\mathrm{P}(B) = (1/6)\cdot 0.3 + (1/6)\cdot 0.7 = 1/6.
\end{aligned}$$

2. Alice tosses a six-sided dice, then she tosses R fair coins, where R is roll of the die. Given that all the coin tosses came out tails, find the probabilities of each outcome for the die.

   Solution: Let Y_i be the event that the roll of the die is i, and A be the event that all the coin tosses are tails. Then $\mathrm{P}(A|Y_i) = (1/2)^i$. By Bayes' rule,

$$\mathrm{P}(Y_i|A) = \frac{\mathrm{P}(A|Y_i)\cdot \mathrm{P}(Y_i)}{\mathrm{P}(A)} = \left(\frac{1}{2}\right)^i \cdot \frac{1/6}{\mathrm{P}(A)}.$$

   Since these probabilities must add up to one, it must be that

$$\mathrm{P}(Y_i|A) = \frac{(1/2)^i}{(1/2)^1 + (1/2)^2 + \cdots + (1/2)^6},$$

   so $\mathrm{P}(Y_1|A) = 32/63$, $\mathrm{P}(Y_2|A) = 16/63$, $\mathrm{P}(Y_3|A) = 8/63$, $\mathrm{P}(Y_4|A) = 4/63$, $\mathrm{P}(Y_5|A) = 2/63$, and $\mathrm{P}(Y_6|A) = 1/63$.

3. There are 6 red balls and 1 blue ball. Each ball is randomly placed in one of two bins.

   (a) What is the probability that the bin with the larger number of balls contains k balls ($k \in \{4, 5, 6, 7\}$)?

   (b) What is the probability that the bin with the larger number of balls contains the blue ball? (*Hint:* Use Bayes' rule.)

   Solution: (a) The sample space Ω consists of all sequences of length 7, where the value of each position can be either `1` or `2`, denoting which bin the ball goes to. Ω has size 2^7. Let E_k denote the event the bin with the larger number of balls contains k balls. Then E_k consists of strings that contain k `1`s and $7 - k$ `2`s, or k `2`s and $7 - k$ `1`s. Therefore $|E_k| = (\binom{7}{k} + \binom{7}{7-k})/2^7 = \binom{7}{k}/2^6$. For $k \in \{4, 5, 6, 7\}$, we have

k	4	5	6	7
$P(E_k)$	35/64	21/64	7/64	1/64

(b) Let B denote the event the larger bin contains the blue ball. We have $P(B \mid E_k) = k/7$. Using the rule of average conditional probabilities, we have

$$P(B) = P(E_4)P(B \mid E_4) + P(E_5)P(B \mid E_5) + P(E_6)P(B \mid E_6) + P(E_7)P(B \mid E_7).$$

Plugging in the values we get $P(B) = 21/32$.

4. Jar A contains 10 black balls and jar B contains 10 white balls. At each step, a ball is picked at random from each jar and moved to the other jar (so the number of balls in each jar stays the same). What is the probability that after four steps the initial configuration is recovered? *(Textbook problem 1.23)*

   Solution: Let $p_t(i)$ be the probability that after t steps, jar A contains i white balls. We want to know $p_4(0)$. By the total probability theorem,

   $$p_4(0) = (1/10)^2 \cdot p_3(1)$$

   because conditioned on jar A having exactly one white after 3 steps, the initial configuration is recovered with probability $(1/10)^2$, and otherwise the initial configuration cannot be recovered. By the same reasoning,

   $$p_3(1) = (2/10)^2 \cdot p_2(2) + 2 \cdot (9/10) \cdot (1/10) \cdot p_2(1) + 1 \cdot p_2(0).$$

   After one step, jar A must have exactly one white ball, so

   $$p_2(0) = (1/10)^2, \qquad p_2(1) = 2 \cdot (9/10) \cdot (1/10), \qquad p_2(2) = (9/10)^2$$

   Plugging in we get that $p_3(1) = 0.0748$ and so $p_4(0) = 0.000748$.

5. If Alice flips 10 coins and Bob flips 9 coins, what is the probability that Alice gets more heads than Bob? (*Hint:* Use conditioning.)

   Solution: The sample space consists of all sequences of 19 `H`s and `T`s, where the first 9 elements in the sequence denote the outcomes of Bob's flips and the next 10 elements denote the outcomes of Alice's flips. We assume equally likely outcomes.

   Let E be the event that Alice gets more heads than Bob. After Alice and Bob have both flipped their first nine coins, there are three possibilities: Alice gets more heads, Bob gets more heads, or there is a tie: Alice and Bob get the same number of heads. Let A, B and T denote those three events respectively. The events A, B, and T partition the sample space. By the rule of average conditional probabilities, we have

   $$P(E) = P(E \mid A)P(A) + P(E \mid B)P(B) + P(E \mid T)P(T).$$

   We now calculate the quantities on the right hand side. Given that event A happens, event E happens with certainty, so $P(E \mid A) = 1$. Similarly, given that event B happens, event E is impossible, so $P(E \mid B) = 0$. Given that event T happens, the game is decided by Alice's last coin toss. Since the event of Alice getting a head in her last toss is independent of the outcomes of the previous tosses, $P(E \mid T) = 1/2$. Therefore

   $$P(E) = P(A) + \frac{1}{2}P(T) = P(A) + \frac{1}{2}(1 - P(A) - P(B)) = \frac{1}{2} + \frac{1}{2}P(A) - \frac{1}{2}P(B).$$

   By symmetry, $P(A) = P(B)$, and so $P(E) = 1/2$.