Examples on Pumping Lemma and Minimization of DFA

Zhao Qiao
qzhao@cse.cuhk.edu.hk

Department of Computer Science & Engineering
The Chinese University of Hong Kong

September 28, 2008

Outline

1. Pumping Lemma?
   - Adversary Argument
   - Explanation
   - Examples
2. Regular or not?
   - General Method
   - Examples
3. Minimization of FA
   - Example

Adversary Argument

Pumping Lemma

$$L \text{ is regular} \Rightarrow (\exists n)(\forall z)\Big(z \in L, |z| \geq n \Rightarrow$$

$$(\exists u, v, w)\big((z = uvw, |uv| \leq n, |v| \geq 1) \textit{ and } (\forall i)uv^iw \in L\big)\Big)$$

$$\Updownarrow$$

Adversary Argument

$$L \text{ is not regular} \Leftarrow (\forall n)(\exists z)\Big(z \in L, |z| \geq n,$$

$$(\forall u, v, w)\big((z = uvw, |uv| \leq n, |v| \geq 1) \Rightarrow (\exists i)uv^iw \notin L\big)\Big)$$

Explanation

Using the adversary argument,we can verify a non-regular language L by the following game:

Game Proof

- the adversary pick an arbitrary n to challenge us for a string z.
- ? we construct a special string z in L with length greater than or equal to n.
- the adversary arbitrarily break z into u, v and w,where v is not empty and uv's length less or equal to n.
- ? if we can always choose a i to show him that uv^iw is not in L,then we win.

Palindromes over $\{a, b\}$

$\{ww^R | w \in \{a, b\}^*\}$

- the adversary pick an arbitrary n to challenge us for a string z.

Palindromes over $\{a, b\}$

$\{ww^R | w \in \{a, b\}^*\}$

- the adversary pick an arbitrary n to challenge us for a string z.
- ☆? How to choose z in L?The following moves will mess with the first n symbols of our z,and we have to make sure the outcome is not in L.

Palindromes over $\{a, b\}$

$\{ww^R | w \in \{a, b\}^*\}$

- the adversary pick an arbitrary n to challenge us for a string z.
- we choose $z = a^n bba^n$

Palindromes over $\{a, b\}$

$\{ww^R | w \in \{a,b\}^*\}$

- the adversary pick an arbitrary n to challenge us for a string z.
- we choose $z = a^n bba^n$
- the adversary arbitrarily break z into u, v and w,where v is not empty and uv's length less than or equal to n.

Palindromes over $\{a, b\}$

$\{ww^R | w \in \{a, b\}^*\}$

- ✪ the adversary pick an arbitrary n to challenge us for a string z.
- ☆ we choose $z = a^n bba^n$
- ✪ the adversary arbitrarily break z into u, v and w,where v is not empty and uv's length less than or equal to n.
- ☆ u,v only contain a; w contains a trailing substring bba^n,and maybe some leading a's.If we set $i = 0$(pump v out),then $uv^i w = uw$ will have less leading a's than its trailing a's,so uw is not a palindrome.✌

Palindromes over $\{a, b\}$

$\{ww^R | w \in \{a, b\}^*\}$

- the adversary pick an arbitrary n to challenge us for a string z.
- we choose $z = a^n bba^n$
- the adversary arbitrarily break z into u, v and w,where v is not empty and uv's length less than or equal to n.
- u,v only contain a; w contains a trailing substring bba^n,and maybe some leading a's.If we set $i = 0$(pump v out),then $uv^i w = uw$ will have less leading a's than its trailing a's,so uw is not a palindrome.✌
- In fact,we can choose any i other than 1.

Twin strings over over $\{a, b\}$

$\{ww | w \in \{a, b\}^*\}$

- the adversary pick an arbitrary n to challenge us for a string z.

Twin strings over over $\{a, b\}$

$\{ww|w \in \{a, b\}^*\}$

- the adversary pick an arbitrary n to challenge us for a string z.
- we choose $z = a^n b a^n b$

Twin strings over over $\{a, b\}$

$\{ww|w \in \{a, b\}^*\}$

- the adversary pick an arbitrary n to challenge us for a string z.
- we choose $z = a^n b a^n b$
- the adversary arbitrarily break z into u, v and w,where v is not empty and uv's length less than or equal to n.

Twin strings over over $\{a, b\}$

$\{ww | w \in \{a, b\}^*\}$

- the adversary pick an arbitrary n to challenge us for a string z.
- we choose $z = a^n b a^n b$
- the adversary arbitrarily break z into u, v and w,where v is not empty and uv's length less than or equal to n.
- u,v only contain a; w contains a trailing substring $ba^n b$,and maybe some leading a's.If we set $i = 0$(pump v out),then $uv^i w = uw$ will have less leading a's before the first b than its a's between 2 b's,so uw is not a twin string.✌

Twin strings over over $\{a, b\}$

$\{ww | w \in \{a, b\}^*\}$

- ✪ the adversary pick an arbitrary n to challenge us for a string z.
- ☆ we choose $z = a^n ba^n b$
- ✪ the adversary arbitrarily break z into u, v and w,where v is not empty and uv's length less than or equal to n.
- ☆ u,v only contain a; w contains a trailing substring $ba^n b$,and maybe some leading a's.If we set $i = 0$(pump v out),then $uv^i w = uw$ will have less leading a's before the first b than its a's between 2 b's,so uw is not a twin string.✌
- ☛ can we choose other i's to win?

General Method

- To prove a language to be regular,we can use regular expression,DFA,NFA or ε- NFA to construct it directly.

General Method

- To prove a language to be regular,we can use regular expression,DFA,NFA or ε- NFA to construct it directly.
- We can also use the closure properties of regular languages: union,concatenation,Kleene closure, complement,intersection,substitution(quotient).

General Method

- To prove a language to be regular,we can use regular expression,DFA,NFA or ε- NFA to construct it directly.
- We can also use the closure properties of regular languages: union,concatenation,Kleene closure, complement,intersection,substitution(quotient).
- To prove a language to be non-regular,we can use pumping lemma and the closure properties of regular languages.

Q1

L is a regular language over {a,b,c},Deicide whether the following languages are regular.

Problems

a $\{w | w \in L, a \notin w\}$

Hints

Q1

L is a regular language over {a,b,c},Deicide whether the following languages are regular.

Problems

a $\{w|w \in L, a \notin w\}$

Hints

a concatenation & complement

Q1

L is a regular language over {a,b,c},Deicide whether the following languages are regular.

Problems

a $\{w|w \in L, a \notin w\}$

b $\{waw|w \in L\}$

Hints

a concatenation & complement

Q1

L is a regular language over {a,b,c},Deicide whether the following languages are regular.

Problems

a $\{w|w \in L, a \notin w\}$
b $\{waw|w \in L\}$

Hints

a concatenation & complement
b like palindromes→pumping lemma

Q1

L is a regular language over {a,b,c},Deicide whether the following languages are regular.

Problems

a $\{w|w \in L, a \notin w\}$
b $\{waw|w \in L\}$
c $\{uv|u \in L, v \notin L\}$

Hints

a concatenation & complement
b like palindromes→pumping lemma

Q1

L is a regular language over {a,b,c},Deicide whether the following languages are regular.

Problems

a $\{w|w \in L, a \notin w\}$
b $\{waw|w \in L\}$
c $\{uv|u \in L, v \notin L\}$

Hints

a concatenation & complement
b like palindromes→pumping lemma
c concatenation & complement

Q2

Prove that the following languages are non-regular.

Problems

a. all strings over {a, b} with the same number of a's and b's.

Hints

Q2

Prove that the following languages are non-regular.

Problems

a. all strings over {a, b} with the same number of a's and b's.

Hints

a. $a^n b^n$

Q2

Prove that the following languages are non-regular.

Problems

a. all strings over {a, b} with the same number of a's and b's.
b. all strings over (,) in which the parentheses are paired.

Hints

a. $a^n b^n$

Q2

Prove that the following languages are non-regular.

Problems

a. all strings over {a, b} with the same number of a's and b's.
b. all strings over (,) in which the parentheses are paired.

Hints

a. $a^n b^n$
b. $(^n)^n$

Q2

Prove that the following languages are non-regular.

Problems

a. all strings over {a, b} with the same number of a's and b's.
b. all strings over (,) in which the parentheses are paired.
c. all strings over {a, b} in which the number of a's is a perfect cube.

Hints

a. $a^n b^n$
b. $(^n)^n$

Q2

Prove that the following languages are non-regular.

Problems

a. all strings over {a, b} with the same number of a's and b's.
b. all strings over (,) in which the parentheses are paired.
c. all strings over {a, b} in which the number of a's is a perfect cube.

Hints

a. $a^n b^n$
b. $(^n)^n$
c. $n < (n+1)^3 - n^3$

Q2

Prove that the following languages are non-regular.

Problems

a all strings over {a, b} with the same number of a's and b's.

b all strings over (,) in which the parentheses are paired.

c all strings over {a, b} in which the number of a's is a perfect cube.

d all non-palindromes over {a, b}.

Hints

a $a^n b^n$

b $(^n)^n$

c $n < (n+1)^3 - n^3$

Q2

Prove that the following languages are non-regular.

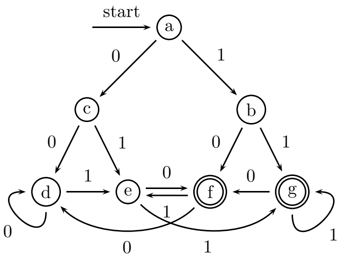
Problems

- a all strings over {a, b} with the same number of a's and b's.
- b all strings over (,) in which the parentheses are paired.
- c all strings over {a, b} in which the number of a's is a perfect cube.
- d all non-palindromes over {a, b}.

Hints

- a $a^n b^n$
- b $(^n)^n$
- c $n < (n+1)^3 - n^3$
- d closure property of complement

Minimization of FA

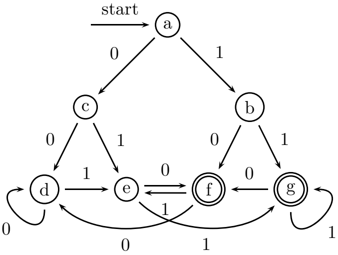


initial mark

	a	b	c	d	e	f
b						
c						
d						
e						
f	X	X	X	X	X	
g	X	X	X	X	X	

mark final and non-final pair

Minimization of FA

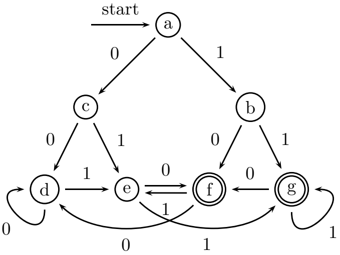


mark ab

	a	b	c	d	e	f
b	X					
c						
d						
e						
f	X	X	X	X	X	
g	X	X	X	X	X	

$$\delta(a,0) = c, \delta(b,0) = f$$

Minimization of FA



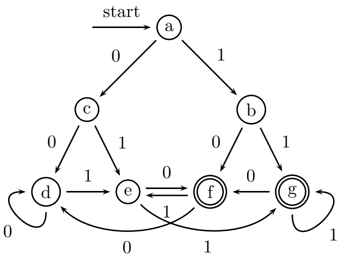
mark ac

	a	b	c	d	e	f
b	X					
c	cd,be					
d						
e						
f	X	X	X	X	X	
g	X	X	X	X	X	

$$\delta(a,0) = c, \delta(c,0) = d$$

$$\delta(a,1) = b, \delta(c,1) = e$$

Minimization of FA



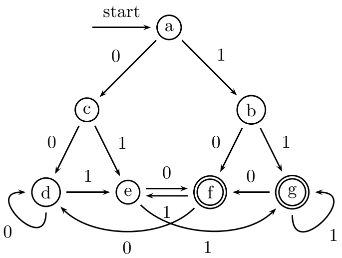
mark ad

b	X					
c	cd,be					
d	cd,be					
e						
f	X	X	X	X	X	
g	X	X	X	X	X	
	a	b	c	d	e	f

$$\delta(a,0) = c, \delta(d,0) = d$$

$$\delta(a,1) = b, \delta(d,1) = e$$

Minimization of FA

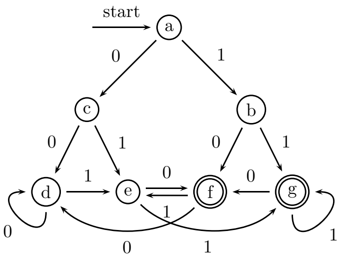


mark ae

b	X					
c	cd,be					
d	cd,be					
e	X					
f	X	X	X	X	X	
g	X	X	X	X	X	
	a	b	c	d	e	f

$$\delta(a,0) = c, \delta(e,0) = f$$

Minimization of FA

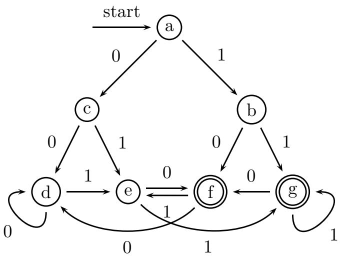


mark bc

	a	b	c	d	e	f
b	X					
c	cd,be	X				
d	cd,be					
e	X					
f	X	X	X	X	X	
g	X	X	X	X	X	

$$\delta(b,0)=f,\delta(c,0)=d$$

Minimization of FA

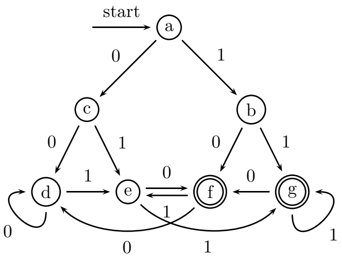


mark bd

	a	b	c	d	e	f
b	X					
c	cd,be	X				
d	cd,be	X				
e	X					
f	X	X	X	X	X	
g	X	X	X	X	X	

$$\delta(b,0) = f, \delta(d,0) = d$$

Minimization of FA



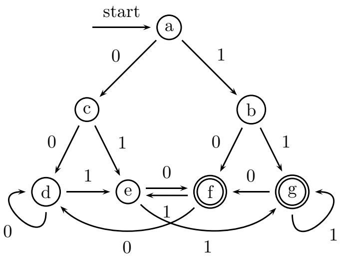
mark be

b	X					
c	cd,be	X				
d	cd,be	X				
e	X	-				
f	X	X	X	X	X	
g	X	X	X	X	X	
	a	b	c	d	e	f

$$\delta(b, 0) = \delta(e, 0) = f$$

$$\delta(b, 1) = \delta(e, 1) = g$$

Minimization of FA



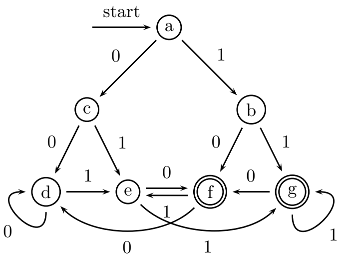
mark dc

b	X					
c	cd,be	X				
d	cd,be	X	-			
e	X	-				
f	X	X	X	X	X	
g	X	X	X	X	X	
	a	b	c	d	e	f

$$\delta(d, 0) = \delta(c, 0) = d$$

$$\delta(d, 1) = \delta(c, 1) = e$$

Minimization of FA

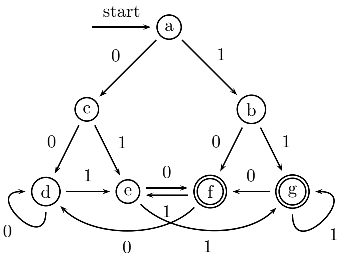


mark ec

b	X					
c	cd,be	X				
d	cd,be	X	-			
e	X	-	X			
f	X	X	X	X	X	
g	X	X	X	X	X	
	a	b	c	d	e	f

$$\delta(e,0) = f, \delta(c,0) = d$$

Minimization of FA

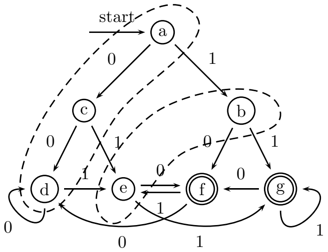


mark de

b	X					
c	cd,be	X				
d	cd,be	X	-			
e	X	-	X	X		
f	X	X	X	X	X	
g	X	X	X	X	X	
	a	b	c	d	e	f

$$\delta(d,0) = d, \delta(e,0) = f$$

Minimization of FA

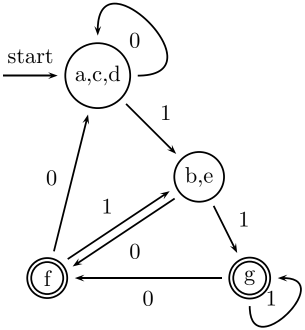


mark fg

b	X					
c	cd,be	X				
d	cd,be	X	-			
e	X	-	X	X		
f	X	X	X	X	X	
g	X	X	X	X	X	X
	a	b	c	d	e	f

$$\delta(f,0) = d, \delta(g,0) = f$$

Minimization of FA



merge non-distingushable states

b	X					
c	cd,be	X				
d	cd,be	X	-			
e	X	-	X	X		
f	X	X	X	X	X	
g	X	X	X	X	X	X
	a	b	c	d	e	f

merge a,c,d and b,e