1 Area of a Smooth xy-Monotone Surface

Let \( S \) be a smooth xy-monotone surface, and \( D \) the projection of \( S \) onto the xy-plane. Next, we will define its area. Note that this currently still remains undefined—you may have a clear understanding of the area of a two-dimensional region, but we are talking about a three-dimensional surface here.

We need to introduce some notions that will be helpful along the way. Consider an arbitrary point \( q = (x, y) \) in \( D \). This point uniquely corresponds to a point \( p = (x, y, z) \) on \( S \) by the xy-monotonicity of \( S \). Let \( N \) be a normal vector of \( S \) at \( p \), and \( \gamma \) be the angle between the direction of \( N \) and the positive direction of the z-axis (namely, of \( k = [0, 0, 1] \)). Note that \( \gamma \) is a function of \( x, y \), because of which we will denote it by \( \gamma(x, y) \).

We are now ready to define the area \( A \) of \( S \) as the value of a double integral:

\[
A = \int \int_D \frac{1}{|\cos(\gamma(x, y))|} \, dx \, dy.
\] (1)

The above definition is in fact much less mysterious than it may appear. Let \( D' \) be a small region in \( D \) that contains \( q \), and \( S' \) be the part of \( S \) such that \( D' \) is precisely the projection of \( S' \) onto the xy-plane. In the figure above, \( D' \) is shown in green, and \( S' \) in pink. When \( D' \) is sufficiently small, we can regard \( S' \) approximately as a planar region embedded in \( \mathbb{R}^3 \). Thus, \( \text{area}(D') \approx \text{area}(S') \cdot |\cos \gamma| \). Now, imagine partitioning \( D \) into a huge number of such small “green” pieces, and sum up the areas of all the “pink” pieces on \( S \) that correspond to those green pieces, respectively. The sum is an approximation of the area of \( S \). When the number of “green” pieces tends to infinity, the sum equals the value of the double integral in (1).

The key to evaluating (1) is to work out \( \cos(\gamma(x, y)) \). The rest of the lecture notes gives two ways to do so, depending on the representation of \( S \).
2 Area Calculation: Method 1

Suppose that \( S \) is given as an equation \( f(x, y, z) = 0 \). We know that \( \nabla f = \left[ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right] \) is a normal vector at point \( p = (x, y, z) \). We thus know that:

\[
\begin{align*}
\left[ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right] \cdot \mathbf{k} &= \left| \mathbf{k} \right| \cos \gamma \\
\Rightarrow \cos \gamma &= \frac{\left[ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right] \cdot \mathbf{k}}{\left| \mathbf{k} \right|} \\
&= \frac{\frac{\partial f}{\partial z}}{\sqrt{\left( \frac{\partial f}{\partial x} \right)^2 + \left( \frac{\partial f}{\partial y} \right)^2 + \left( \frac{\partial f}{\partial z} \right)^2}}. \tag{2}
\end{align*}
\]

**Example 1.** Calculate the area of the hemisphere \( x^2 + y^2 + z^2 = 1 \) with \( z \geq 0 \).

**Solution.** Let \( S \) be the hemisphere, and \( D \) be the projection of \( S \) onto the xy-plane, namely, the disc \( x^2 + y^2 \leq 1 \).

Introduce \( f(x, y, z) = x^2 + y^2 + z^2 - 1 \). \( S \) can be described by \( f(x, y, z) = 0 \) with \( z \geq 0 \). We thus have: \( \frac{\partial f}{\partial x} = 2x, \frac{\partial f}{\partial y} = 2y, \) and \( \frac{\partial f}{\partial z} = 2z \). By (2), we know:

\[
\cos(\gamma(x, y, z)) = \frac{2z}{\sqrt{(2x)^2 + (2y)^2 + (2z)^2}} = z.
\]

Therefore, from (2), we know that the area of \( S \) equals

\[
\iint_D \frac{1}{z} \, dx \, dy = \iint_D \frac{1}{\sqrt{1 - x^2 - y^2}} \, dx \, dy = 2\pi.
\]

\( \square \)

3 Area Calculation: Method 2

Suppose that \( S \) is given in a parametric form with parameters \( u, v \), namely, the x-, y-, and z-coordinates of each point on \( S \) are given by functions \( x(u, v), y(u, v), \) and \( z(u, v) \), respectively.

Define vectors \( \mathbf{r}_u \) and \( \mathbf{r}_v \) as follows:

\[
\begin{align*}
\mathbf{r}_u &= \left[ \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right] \\
\mathbf{r}_v &= \left[ \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right].
\end{align*}
\]

We have:

**Lemma 1.** If \( \mathbf{r}_u \) and \( \mathbf{r}_v \) neither have the same nor have the opposite directions, then

\[
\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v
\]

is a normal vector of \( S \) at the point \((x(u, v), y(u, v), z(u, v))\).
Proof. Notice that \( r_u \) is a tangent vector of the curve \( r(u) = [x(u,v), y(u,v), z(u,v)] \) by fixing \( v \) to be a constant. Likewise, \( r_v \) is a tangent vector of the curve \( r'(v) = [x(u,v), y(u,v), z(u,v)] \) by fixing \( u \) to be a constant. Hence, \( r_u \) and \( r_v \) determine the tangent plane of \( S \) at the point \( (x(u,v), y(u,v), z(u,v)) \). The lemma then follows from the geometric interpretation of cross product. 

Therefore:

\[
N \cdot k = |N||k| \cos \gamma \\
\Rightarrow \cos \gamma = \frac{N \cdot k}{|N|} \\
\text{(by definition of cross product)} = \frac{\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}}{|N|}.
\]

Next, we convert (1) into a double integral on \( u, v \). For this purpose, we calculate the Jacobian:

\[
J = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}.
\]

Denoting \( R \) as the set of \( (u, v) \) defining \( D \), we know:

\[
\int\int_D \frac{1}{|\cos(\gamma(x,y))|} \, dx\,dy = \int\int_R \frac{|N|}{|N|} \cdot |J| \, du\,dv
\]

\[
= \int\int_R |N| \, du\,dv. \quad (3)
\]

Example 2. Calculate the area of the hemisphere \( x^2 + y^2 + z^2 = 1 \) with \( z \geq 0 \).

Solution. Let \( S \) be the hemisphere, and \( D \) be the projection of \( S \) onto the xy-plane, namely, the disc \( x^2 + y^2 \leq 1 \).

Represent \( S \) as a parametric form:

\[
x(u,v) = \cos u \cdot \sin v \\
y(u,v) = \sin u \cdot \sin v \\
z(u,v) = \cos v
\]

for \( u \in [0, 2\pi] \) and \( v \in [0, \pi/2] \). Let \( R \) be the set of such \( (u,v) \). Therefore:

\[
r_u = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \end{vmatrix} = \begin{bmatrix} -\sin u \sin v, \cos u \sin v, 0 \end{bmatrix} \\
r_v = \begin{vmatrix} \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} = \begin{bmatrix} \cos u \cos v, \sin u \cos v, -\sin v \end{bmatrix}.
\]

Hence:

\[
N = r_u \times r_v \\
= \begin{bmatrix} -\cos u \sin^2 v, -\sin u \sin^2 v, -\sin^2 u \sin v \cos v - \cos^2 u \sin v \cos v \end{bmatrix} \\
= \begin{bmatrix} -\cos u \sin^2 v, -\sin u \sin^2 v, -\sin v \cos v \end{bmatrix}
\]
which means

\[ |N| = \sqrt{\sin^4 v + \sin^2 v \cos^2 v} = \sin v. \]

Therefore, the area of the hemisphere is

\[
\iint_R |N| \, du \, dv = \iint_R \sin v \, du \, dv \\
= \int_0^{2\pi} \left( \int_0^{\pi/2} \sin v \, dv \right) \, du = 2\pi.
\]