Exercises: Matrix Rank

Problem 1. Calculate the rank of the following matrix:

\[
\begin{bmatrix}
0 & 16 & 8 & 4 \\
2 & 4 & 8 & 16 \\
16 & 8 & 4 & 2 \\
4 & 8 & 16 & 2 \\
\end{bmatrix}
\]

Solution. To compute the rank of a matrix, remember two key points: (i) the rank does not change under elementary row operations; (ii) the rank of a row-echelon matrix is easy to acquire. Motivated by this, we convert the given matrix into row echelon form using elementary row operations:

\[
\begin{bmatrix}
0 & 16 & 8 & 4 \\
2 & 4 & 8 & 16 \\
16 & 8 & 4 & 2 \\
4 & 8 & 16 & 2 \\
\end{bmatrix} \Rightarrow \begin{bmatrix}
2 & 4 & 8 & 16 \\
16 & 8 & 4 & 2 \\
4 & 8 & 16 & 2 \\
0 & 16 & 8 & 4 \\
\end{bmatrix} \Rightarrow \begin{bmatrix}
1 & 2 & 4 & 8 \\
0 & -24 & -60 & -126 \\
0 & 0 & 0 & -30 \\
0 & 4 & 2 & 1 \\
\end{bmatrix} \Rightarrow \begin{bmatrix}
1 & 2 & 4 & 8 \\
0 & 4 & 2 & 1 \\
0 & -24 & -60 & -126 \\
0 & 0 & 0 & -30 \\
\end{bmatrix} \Rightarrow \begin{bmatrix}
1 & 2 & 4 & 8 \\
0 & 4 & 2 & 1 \\
0 & 0 & -48 & -120 \\
0 & 0 & 0 & -30 \\
\end{bmatrix}
\]

As this matrix has 4 non-zero rows, we conclude that the original matrix has rank 4.

Problem 2. Calculate the rank of the following matrix:

\[
\begin{bmatrix}
4 & -6 & 0 \\
-6 & 0 & 1 \\
0 & 9 & -1 \\
0 & 1 & 4 \\
\end{bmatrix}
\]
Solution.

\[
\begin{bmatrix}
4 & -6 & 0 \\
-6 & 0 & 1 \\
0 & 9 & -1 \\
0 & 1 & 4
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
2 & -3 & 0 \\
-6 & 0 & 1 \\
0 & 9 & -1 \\
0 & 1 & 4
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
2 & -3 & 0 \\
0 & -9 & 1 \\
0 & 9 & -1 \\
0 & 1 & 4
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
2 & -3 & 0 \\
0 & -9 & 1 \\
0 & 0 & 0 \\
0 & 0 & 37/9
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
2 & -3 & 0 \\
0 & -9 & 1 \\
0 & 0 & 37/9 \\
0 & 0 & 0
\end{bmatrix}
\]

Hence, the rank of the original matrix is 3.

**Problem 3.** Judge whether the following vectors are linearly independent.

\[
\begin{bmatrix}
3 \\
6 \\
12 \\
6 \\
9
\end{bmatrix},
\begin{bmatrix}
0 \\
1 \\
1 \\
0 \\
0
\end{bmatrix},
\begin{bmatrix}
1 \\
0 \\
2 \\
2 \\
1
\end{bmatrix},
\begin{bmatrix}
2 \\
4 \\
4 \\
0 \\
2
\end{bmatrix},
\begin{bmatrix}
1 \\
2 \\
1 \\
2 \\
2
\end{bmatrix},
\begin{bmatrix}
2 \\
4 \\
4 \\
0 \\
1
\end{bmatrix}
\]

If they are not, find the largest number of linearly independent vectors among them.

**Solution.** This question is essentially asking for the rank of matrix:

\[
\begin{bmatrix}
3 & 0 & 1 & 2 \\
6 & 1 & 0 & 0 \\
12 & 1 & 2 & 4 \\
6 & 0 & 2 & 4 \\
9 & 0 & 1 & 2
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
3 & 0 & 1 & 2 \\
0 & 1 & -2 & -4 \\
0 & 1 & -2 & -4 \\
0 & 0 & 0 & 0 \\
0 & 0 & -2 & -4 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
3 & 0 & 1 & 2 \\
0 & 1 & -2 & -4 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

The rank of the matrix is 3. This means that the maximum number of linearly independent vectors is 3. They are the ones that correspond to the non-zero rows of the final matrix:

\[
\begin{bmatrix}
3, 0, 1, 2 \\
6, 1, 0, 0 \\
9, 0, 1, 2
\end{bmatrix}
\]
Problem 4. Prove: if $A$ is not square, then either the row vectors or the column vectors are linearly dependent.

Proof. The maximum number of linearly independent row vectors is the rank of $A$, while the maximum number of linearly independent column vectors is the rank of $A^T$. Suppose that $A$ is an $m \times n$ matrix. If $m < n$, then $\text{rank } A^T = \text{rank } A \leq m < n$. Therefore, the column vectors are linearly dependent. Similarly, if $n < m$, then the row vectors are linearly dependent.

Problem 5. Let $S$ be an arbitrary set of vectors in $\mathbb{R}^3$. Prove that there are at most 3 linearly independent vectors in $S$.

Proof. Let $n$ be the number of vectors in $S$. For an $n \times 3$ matrix $A$ where the $i$-th $(1 \leq i \leq n)$ row is the $i$-th vector in $S$. Clearly, $\text{rank } A = \text{rank } A^T \leq 3$. Hence, $S$ can have at most 3 linearly independent vectors.

Problem 6 (Hard). Prove: $\text{rank}(AB) \leq \text{rank} A$.

Proof. Suppose that $A$ is an $m \times n$ matrix, and $B$ an $n \times p$ matrix. Let $d = \text{rank } A$. Without loss of generality, assume that the first $d$ rows of $A$ are linearly independent. Denote the row vectors of $A$ as $r_1, ..., r_m$ in top down order, and the column vectors of $B$ as $c_1, ..., c_p$ in left-to-right order.

We will prove that for any $i \in [d+1, m]$, the $i$-th row of $AB$ is a linear combination of the first $d$ rows of $AB$. This, in effect, shows that $\text{rank}(AB) \leq d$.

We know that the first $d$ rows of $AB$ are:

$$v_1 = [r_1 \cdot c_1, r_1 \cdot c_2, ..., r_1 \cdot c_p]$$
$$v_2 = [r_2 \cdot c_1, r_2 \cdot c_2, ..., r_2 \cdot c_p]$$
$$\vdots$$
$$v_d = [r_d \cdot c_1, r_d \cdot c_2, ..., r_d \cdot c_p]$$

while the $i$-th ($i \in [d+1, m]$) row of $AB$ is:

$$v_i = [r_i \cdot c_1, r_i \cdot c_2, ..., r_i \cdot c_p]$$

Since $r_i$ is a linear combination of $r_1, r_2, ..., r_d$, there exist real values $\alpha_1, ..., \alpha_d$ that (i) are not all zero, and (ii) satisfy:

$$r_i = \sum_{z=1}^{d} \alpha_z r_z$$

This means that for any $j \in [1, p]$, we have

$$r_i \cdot c_j = \sum_{z=1}^{d} \alpha_z (r_z \cdot c_j)$$

This, in turn, indicates that

$$v_i = \sum_{z=1}^{d} \alpha_z v_z$$
namely, $v_i$ is a linear combination of $v_1, \ldots, v_d$. \hfill \square

**Problem 7 (Very Hard).** Prove: $\text{rank}(A + B) \leq \text{rank } A + \text{rank } B$.

**Proof.** Let $A, B$ be $m \times n$ matrices. Construct an $(2m) \times (2n)$ matrix:

$$Q = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

$\text{rank } Q = \text{rank } A + \text{rank } B$ (you can see this by converting $Q$ into row-echelon form).

Also observe that $Q$ has the same rank as

$$\begin{bmatrix} A & 0 \\ A & B \end{bmatrix}$$

which has the same rank as

$$\begin{bmatrix} A & A \\ A & A + B \end{bmatrix}$$

Since the rank of a submatrix cannot exceed the rank of the whole matrix, we know that $\text{rank } (A + B)$ is at most the rank of $Q$, which as mentioned earlier is $\text{rank } A + \text{rank } B$. \hfill \square