Problem 1

Show $E_{TM} = \{ M : M \text{ is a TM and } L(M) = \emptyset \}$ is undecidable.

Solution

We have already known that $L = \{ (w, M) : w \text{ is accepted by } M \}$ is undecidable. Suppose $E_{TM}$ is decidable, then there exists a TM, say $A$, that can decide $E_{TM}$. For any input $(w, M)$, construct $M'$ as follows.

(i) Rejects if the input does not equal to $w$.
(ii) If the input equals to $w$, simulate $M$ on $w$. Accepts if $M$ rejects $w$ and rejects if $M$ accepts $w$.

Now, we show that $A$ accepts $M'$ iff $M$ accepts $w$. $\Rightarrow$: If $A$ accepts $M'$, then $M'$ rejects all strings including $w$ and this implies $M$ accepts $w$. $\Leftarrow$: If $M$ accepts $w$, then $M'$ rejects $w$. Since $M'$ also rejects all the other strings, $M'$ is accepted by $A$.

Now, we construct a TM for $L$. On input $(w, M)$, construct $M'$ and run $A$ on it, accepts if and only if $A$ accepts. This contradict to the fact that $L$ is undecidable.

Problem 2

$EQ = \{ (M_1, M_2) : M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2) \}$ is undecidable.

Solution

Suppose $EQ$ is decidable, then there exists a TM, say $A$, that can decide $EQ$. Now, we use $A$ to construct a TM to decide $L$, which is a contradiction. $M_1$ rejects all strings. For input $(w, M)$, construct $M_2$ as follows,

(i) Rejects if the input does not equal to $w$.
(ii) If the input equals to $w$, simulate $M$ on $w$. Accepts if $M$ rejects $w$ and rejects if $M$ accepts $w$.

Since the definition of $M_2$ is the same as that of $M'$ in problem 1, then $L(M_2) = \emptyset$ is equivalent to $M$ accepts $w$, and $M_2$ rejects all strings is trivially equivalent to $L(M_2) = L(M_1)$.

Now, we construct a TM for $L$. On input $(w, M)$, construct $(M_1, M_2)$ and run $A$ on it, accepts if and only if $A$ accepts. This contradict to the fact that $L$ is undecidable.
**Problem 3**

Show $T = \{ M : M \text{ is a TM that accepts } w^R \text{ whenever it accepts } w \}$ is undecidable.

**Solution**

We have already known that $L = \{ (w, M) : w \text{ is accepted by } M. \}$ is undecidable. Suppose $T$ is decidable, then there exists a TM, say $A$, that can decide $T$. For any input $(w, M)$, construct $M'$ as follows.

If $w = w^R$, simulate $M$ on $w$. The alphabet set of $M$ is $\Sigma$, without loss of generality, say $a, b \notin \Sigma$. Let $\Sigma \cup \{a, b\}$ be the alphabet set of $M'$. $M'$ rejects all the other strings other than $ab$, for input $ab$, simulate $M$ on $w$,

(i) If $M$ accepts $w$, $M'$ rejects.

(ii) If $M$ rejects $w$, $M'$ accepts.

Now, we show that $A$ accepts $M'$ iff $M$ accepts $w$. "$\Rightarrow$": If $A$ accepts $M'$, since $M'$ rejects all the other strings including $ba$, then $M'$ rejects $ab$ and this implies $M$ accepts $w$. "$\Leftarrow$": If $M$ accepts $w$, then $M'$ rejects $ab$. Since $M'$ rejects all the other strings, $M'$ is accepted by $A$.

Now, we have constructed a TM for $L$. On input $(w, M)$, construct $M'$ and run $A$ on it, accepts if and only if $A$ accepts. This contradict to the fact that $L$ is undecidable.