These are the examples for Tutorial 3 with solutions. The alphabet is $\Sigma = \{0, 1\}$ in all the examples.

**Problem**

Which of these languages is regular?

(a) $L_1 = \{0^m1^n : m > n \geq 0\}$

(b) $L_2 = \{0^{2n} : n \geq 1\}$

(c) $L_3 = \{0^m1^n0^{m+n} : m \geq 1 \text{ and } n \geq 1\}$

(d) $L_4 = \{x : x \text{ does not have three consecutive } 0\text{s}\}$

(e) $L_5 = \{x : x \text{ has an equal number of } 0\text{s and } 1\text{s}\}$

(f) $L_6 = \{x : x = x^R\}$. Recall that $x^R$ is $x$ written backwards; for example, $(011)^R = 110$

(g) $L_7 = \{0^{n^2} : n \text{ is an integer and } n \geq 0\}$

(h) $L_8 = \{0^n : n \text{ is a prime}\}$

(i) $L_9 = \{x : x \text{ has a different number of } 0\text{s and } 1\text{s}\}$

The solutions are on the next page.
Solution

(a) We show $L_1$ is not regular using the pumping lemma. Suppose $L_1$ is regular. Let $n$ be its pumping length. Take $z = 0^n1^{n-1}$, which is in $L_1$. Then $u$ and $v$ consist only of zeros. By the pumping lemma, we can write $z = uvw$ where $|uv| \leq n$ and $|v| \geq 1$ so that $uv^iw \in L_1$ for every $i$. In particular $uw = uv^0w$ should by in $L_1$. But $uw$ has at most $n - 1$ zeros and at least $n - 1$ ones, so $uw \not\in L_1$, a contradiction.

(b) $L_2$ is described by the regular expression $(00)^*$, so it is regular.

(c) We show $L_3$ is not regular using the pumping lemma. Suppose $L_3$ is regular. Let $n$ be its pumping length. Take $z = 0^n1^n0^n2^n$, which is in $L_3$. Then $u$ and $v$ consist only of zeros. By the pumping lemma, we can write $z = uvw$ where $|uv| \leq n$ and $|v| \geq 1$ so that $uv^iw \in L_3$ for every $i$. In particular $uw = uv^0w$ should by in $L_3$. But $uw$ has fewer 0s in the first block than 1s in the second block, so it is not in $L_3$, a contradiction.

(d) The complement of $L_4$ is the language \{ $x$: $x$ contains three consecutive 0s \}. This language is described by the regular expression $(0 + 1)^*000(0 + 1)^*$, so it is regular. Therefore $L_4$ is also regular.

(e) We show $L_5$ is not regular using the pumping lemma. Suppose $L_5$ is regular. Let $n$ be its pumping length. Take $z = 0^n1^n$, which is in $L_5$. Then $u$ and $v$ consist only of zeros. By the pumping lemma, we can write $z = uvw$ where $|uv| \leq n$ and $|v| \geq 1$ so that $uv^iw \in L_5$ for every $i$. In particular $uw = uv^0w$ should by in $L_5$. But $uw$ has fewer 0s in the first block than 1s in the second block, so it is not in $L_5$, a contradiction.

(f) We show $L_6$ is not regular using the pumping lemma. Suppose $L_6$ is regular. Let $n$ be its pumping length. Take $z = 0^n10^n$, which is in $L_6$. Then $u$ and $v$ consist only of zeros. By the pumping lemma, we can write $z = uvw$ where $|uv| \leq n$ and $|v| \geq 1$ so that $uv^iw \in L_6$ for every $i$. In particular $uw = uv^0w$ should by in $L_6$. But $uw$ has the form $0^m10^n$, where $m < n$. So $(uv)^R = 0^n10^m \neq uw$, and $uw$ is not in $L_6$, a contradiction.

(g) We show $L_7$ is not regular using the pumping lemma. Suppose $L_7$ is regular. Let $n$ be its pumping length. Take $z = 0^n3$, which is in $L_7$. By the pumping lemma, we can write $z = uvw$ where $|uv| \leq n$ and $|v| \geq 1$ so that $uv^iw \in L_7$ for every $i$. In particular $uv^2w$ should by in $L_7$. But $uv^2w$ has length $n^2 + |v| \leq n^2 + n$, which is a number strictly between $n^2$ and $(n + 1)^2$ (because $(n + 1)^2 = n^2 + 2n + 1$), so it is not the square of any number. Therefore $uv^2w$ is not in $L_7$, a contradiction.

(h) We show $L_8$ is not regular using the pumping lemma. Suppose $L_8$ is regular. Let $n$ be its pumping length. Take $z = 0^p$, where $p$ is any prime bigger than $n$. (Since there are infinitely many prime numbers, we can always choose such a $p$.) By the pumping lemma, we can write $z = uvw$ where $|uv| \leq n$ and $|v| \geq 1$ so that $uv^iw \in L_8$ for every $i$. Take $i = p + 1$. Then $uw$ has length $p - |v|$ and $v^i$ has length $i|v|$. So $uv^iw$ has length $(p - |v|) + i|v| = (p - |v|) + (p - 1)|v| = p|v| + 1$, which is a product of two numbers greater than one. The length of $uv^iw$ is not a prime number, so $uv^iw \not\in L_8$, a contradiction.
(i) The easier way to prove $L_9$ is not regular goes like this. Suppose it is regular, then $L_5$ is $L_9$’s complement, hence $L_5$ is regular. This contradicts part (e).

If you want to prove $L_9$ is not regular using the pumping lemma, it is also possible, but a bit more difficult. Suppose it is regular and let $n$ be its pumping length. Take $z = 0^n1^{n+n!}$, which is in $L_9$. ($n!$ is the factorial of $n$, given by $n! = 1 \cdot 2 \cdot 3 \ldots n$.) By the pumping lemma, we can write $z = uvw$ where $|uv| \leq n$ and $|v| \geq 1$ so that $uv^i w \in L_9$ for every $i$. But if we set $i = n!/|v| + 1$ (which is an integer because $|v| \leq n$, and so it divides $n!$), we get that $uv^i w$ has $n + (i - 1)|v|$ zeros and $n!$ ones. By our choice of $i$, $uv^i w = 0^{n!}1^{n!} \notin L_9$, a contradiction.