Outline

1. Pumping Lemma?
   - Adversary Argument
   - Explanation
   - Examples

2. Regular or not?
   - General Method
   - Examples

3. Minimization of FA
   - Example
Pumping Lemma

$L$ is regular $\Rightarrow$ $(\exists n)(\forall z)\left( z \in L, |z| \geq n \Rightarrow (\exists u, v, w)(z = uvw, |uv| \leq n, |v| \geq 1) \text{ and } (\forall i)uv^iw \in L \right)$

$\Leftrightarrow$

Adversary Argument

$L$ is not regular $\Leftarrow$ $(\forall n)(\exists z)\left( z \in L, |z| \geq n, (\forall u, v, w)((z = uvw, |uv| \leq n, |v| \geq 1) \Rightarrow (\exists i)uv^iw \notin L) \right)$
Using the adversary argument, we can verify a non-regular language $L$ by the following game:

**Game Proof**

- the adversary pick an arbitrary $n$ to challenge us for a string $z$.
- we construct a special string $z$ in $L$ with length greater than or equal to $n$.
- the adversary arbitrarily break $z$ into $u$, $v$ and $w$, where $v$ is not empty and $uv$’s length less or equal to $n$.
- if we can always choose a $i$ to show him that $uv^i w$ is not in $L$, then we win.
Palindromes over \( \{a, b\} \)

\[ \{ww^R|w \in \{a, b\}^*\} \]

- The adversary pick an arbitrary \( n \) to challenge us for a string \( z \).
Palindromes over \( \{a, b\} \)

\[ \{ww^R | w \in \{a, b\}^* \} \]

- ⭐ the adversary pick an arbitrary \( n \) to challenge us for a string \( z \).
- ✳️? How to choose \( z \) in \( L \)? The following moves will mess with the first \( n \) symbols of our \( z \), and we have to make sure the outcome is not in \( L \).
Palindromes over \( \{a, b\} \)

\[ \{ww^R \mid w \in \{a, b\}^*\} \]

- the adversary pick an arbitrary \( n \) to challenge us for a string \( z \).
- we choose \( z = a^n bba^n \)
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- the adversary pick an arbitrary \( n \) to challenge us for a string \( z \).
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- the adversary arbitrarily break \( z \) into \( u, v \) and \( w \), where \( v \) is not empty and \( uv \)'s length less than or equal to \( n \).
- \( u, v \) only contain \( a \); \( w \) contains a trailing substring \( bba^n \), and maybe some leading \( a \)'s. If we set \( i = 0 \) (pump \( v \) out), then \( uv^i w = uw \) will have less leading \( a \)'s than its trailing \( a \)'s, so \( uw \) is not a palindrome.

Zhao Qiao  qzhao@cse.cuhk.edu.hk  Examples on Pumping Lemma and Minimization of DFA
Palindromes over \( \{a, b\} \)

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- The adversary pick an arbitrary \( n \) to challenge us for a string \( z \).
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- In fact, we can choose any \( i \) other than 1.
Twin strings over over \( \{a, b\} \)

\[
\{ww \mid w \in \{a, b\}^*\}
\]

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Twin strings over over \{a, b\}

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Twin strings over over \( \{a, b\} \)

\( \{ww \mid w \in \{a, b\}^*\} \)

- The adversary pick an arbitrary \( n \) to challenge us for a string \( z \).
- We choose \( z = a^n ba^n b \)
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- \( u, v \) only contain \( a \); \( w \) contains a trailing substring \( ba^n b \), and maybe some leading \( a \)'s. If we set \( i = 0 \) (pump \( v \) out), then \( uv^i w = uw \) will have less leading \( a \)'s before the first \( b \) than its \( a \)'s between 2 \( b \)'s, so \( uw \) is not a twin string.
- Can we choose other \( i \)'s to win?
To prove a language to be regular, we can use regular expression, DFA, NFA or $\varepsilon$-NFA to construct it directly.
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We can also use the closure properties of regular languages: union, concatenation, Kleene closure, complement, intersection, substitution (quotient).
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We can also use the closure properties of regular languages: union, concatenation, Kleene closure, complement, intersection, substitution (quotient).

To prove a language to be non-regular, we can use pumping lemma and the closure properties of regular languages.
L is a regular language over \{a,b,c\}, decide whether the following languages are regular.

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<th>Problems</th>
<th>Hints</th>
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<td>w ∈ L, a ∉ w}</td>
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L is a regular language over \{a,b,c\}, decide whether the following languages are regular.

Problems

- a \{w | w \in L, a \notin w \}

Hints

- a concatenation & complement
Q1

L is a regular language over \{a,b,c\}, Decide whether the following languages are regular.

Problems

a \( \{w \mid w \in L, a \notin w\} \)

b \( \{waw \mid w \in L\} \)

Hints

a concatenation & complement
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b like palindromes → pumping lemma
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- b \( \{waw | w \in L\} \)
- c \( \{uv | u \in L, v \notin L\} \)

**Hints**

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Problems

a \(\{w \mid w \in L, \text{ } a \notin w\}\)
b \(\{waw \mid w \in L\}\)
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Hints

a concatenation & complement
b like palindromes → pumping lemma
c concatenation & complement
Prove that the following languages are non-regular.

**Problems**

a. all strings over \{a, b\} with the same number of a’s and b’s.

**Hints**
Prove that the following languages are non-regular.

Problems

- all strings over \{a, b\} with the same number of a’s and b’s.

Hints

- \(a^n b^n\)
Prove that the following languages are non-regular.

**Problems**

a. all strings over \{a, b\} with the same number of a’s and b’s.

b. all strings over (, ) in which the parentheses are paired.

**Hints**

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Prove that the following languages are non-regular.

**Problems**

a. all strings over \{a, b\} with the same number of a’s and b’s.

b. all strings over (, ) in which the parentheses are paired.

**Hints**

a. \(a^n b^n\)

b. \((n)^n\)
Prove that the following languages are non-regular.

Problems

a) all strings over \{a, b\} with the same number of a’s and b’s.

b) all strings over (, ) in which the parentheses are paired.

c) all strings over \{a, b\} in which the number of a’s is a perfect cube.

Hints

a) \(a^n b^n\)

b) \( (n)^n \)
Prove that the following languages are non-regular.

### Problems

- a. all strings over \{a, b\} with the same number of a’s and b’s.
- b. all strings over (, ) in which the parentheses are paired.
- c. all strings over \{a, b\} in which the number of a’s is a perfect cube.

### Hints

- a. \(a^n b^n\)
- b. \((n)^n\)
- c. \(n < (n + 1)^3 - n^3\)
Prove that the following languages are non-regular.

Problems

a. all strings over \{a, b\} with the same number of a’s and b’s.
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c. all strings over \{a, b\} in which the number of a’s is a perfect cube.
d. all non-palindromes over \{a, b\}.

Hints

a. $a^n b^n$
b. $(^n)^n$
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Hints

a. \(a^n b^n\)
b. \((^n)^n\)
c. \(n < (n + 1)^3 - n^3\)
d. closure property of complement
Minimization of FA

initial mark

mark final and non-final pair
Minimization of FA

Examples on Pumping Lemma and Minimization of DFA

\[\delta(a, 0) = c, \delta(b, 0) = f\]
Minimization of FA

**Example**

### Pumping Lemma?

- **Regular or not?**
- **Minimization of FA**

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### Minimization of FA

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- $\delta(a, 0) = c$, $\delta(c, 0) = d$
- $\delta(a, 1) = b$, $\delta(c, 1) = e$

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**Examples on Pumping Lemma and Minimization of DFA**

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Minimization of FA

δ(a, 0) = c, δ(d, 0) = d
δ(a, 1) = b, δ(d, 1) = e
Minimization of FA

\[ \delta(a, 0) = c, \delta(e, 0) = f \]
Minimization of FA

Examples on Pumping Lemma and Minimization of DFA
Examples on Pumping Lemma and Minimization of DFA

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Minimization of FA

mark bd

\[ \delta(b, 0) = f, \delta(d, 0) = d \]
Minimization of FA

\[ \delta(b, 0) = \delta(e, 0) = f \]
\[ \delta(b, 1) = \delta(e, 1) = g \]
Minimization of FA

\[
\begin{align*}
\delta(d, 0) &= \delta(c, 0) = d \\
\delta(d, 1) &= \delta(c, 1) = e
\end{align*}
\]

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Examples on Pumping Lemma and Minimization of DFA
Minimization of FA

\[ \delta(e, 0) = f, \delta(c, 0) = d \]
Minimization of FA

[Diagram of a DFA with labeled transitions and states:]

- Start state: a
- Transitions:
  - a: 0 -> c, 1 -> b
  - c: 0 -> d, 1 -> e
  - e: 0 -> f, 1 -> g
  - d: 0 -> d, 1 -> e
  - f: 0 -> f, 1 -> g

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δ(d, 0) = d, δ(e, 0) = f

Examples on Pumping Lemma and Minimization of DFA

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Minimization of FA

Examples on Pumping Lemma and Minimization of DFA

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\[ \delta(f, 0) = d, \delta(g, 0) = f \]
Minimization of FA

merge non-distinguishable states

merge a,c,d and b,e

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Examples on Pumping Lemma and Minimization of DFA