A NEAR-OPTIMAL PARALLEL ALGORITHM FOR JOINING BINARY RELATIONS

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ABSTRACT. We present a constant-round algorithm in the massively parallel computation (MPC) model for evaluating a natural join where every input relation has two attributes. Our algorithm achieves a load of $\tilde{O}(m/p^{1/\rho})$ where $m$ is the total size of the input relations, $p$ is the number of machines, $\rho$ is the join's fractional edge covering number, and $\tilde{O}(\cdot)$ hides a polylogarithmic factor. The load matches a known lower bound up to a polylogarithmic factor. At the core of the proposed algorithm is a new theorem (which we name the isolated cartesian product theorem) that provides fresh insight into the problem's mathematical structure. Our result implies that the subgraph enumeration problem, where the goal is to report all the occurrences of a constant-sized subgraph pattern, can be settled optimally (up to a polylogarithmic factor) in the MPC model.

1. INTRODUCTION

Understanding the hardness of joins has been a central topic in database theory. Traditional efforts have focused on discovering fast algorithms for processing joins in the random access machine (RAM) model (see [1,5,15–17,20,21] and the references therein). Nowadays, massively parallel systems such as Hadoop [8] and Spark [2] have become the mainstream architecture for analytical tasks on gigantic volumes of data. Direct adaptations of RAM algorithms, which are designed to reduce CPU time, rarely give satisfactory performance on that architecture. In systems like Hadoop and Spark, it is crucial to minimize communication across the participating machines because usually the overhead of message exchanging overwhelms the CPU calculation cost. This has motivated a line of research — which includes this work — that aims to understand the communication complexities of join problems.

Key words and phrases: Joins, Conjunctive Queries, Parallel Computing, Database Theory.
1.1. **Problem Definition.** We will first give a formal definition of the join operation studied in this paper and then elaborate on the computation model assumed.

**Joins.** Let \( \text{att} \) be a countably infinite set where each element is called an *attribute*, and \( \text{dom} \) be another countably infinite set. A *tuple* over a set \( U \subseteq \text{att} \) is a function \( u : U \to \text{dom} \). Given a subset \( V \) of \( U \), define \( u[V] \) as the tuple \( v \) over \( V \) such that \( v(X) = u(X) \) for every \( X \in V \). We say that \( u[V] \) is the *projection* of \( u \) on \( V \).

A *relation* is a set \( R \) of tuples over the same set \( U \) of attributes. We say that the *scheme* of \( R \) is \( U \), and write this fact as \( \text{scheme}(R) = U \). The *arity* of \( R \), denoted as \( \text{arity}(R) \), equals \( |\text{scheme}(R)| \). \( R \) is *unary* or *binary* if \( \text{arity}(R) = 1 \) or \( 2 \), respectively. A value \( x \in \text{dom} \) appears in \( R \) if there exist a tuple \( u \in R \) and an attribute \( X \in U \) such that \( u(X) = x \); we will also use the expression that \( x \) is a value on the attribute \( X \) in \( R \).

A *join query* is a set \( Q \) of relations. Define \( \text{attset}(Q) = \bigcup_{R \in Q} \text{scheme}(R) \). The *result* of the query, denoted as \( \text{Join}(Q) \), is the following relation over \( \text{attset}(Q) \):

\[
\{\text{tuple } u \text{ over } \text{attset}(Q) \mid \forall R \in Q, u[\text{scheme}(R)] \in R\}.
\]

\( Q \) is
- *simple* if no distinct \( R, S \in Q \) satisfy \( \text{scheme}(R) = \text{scheme}(S) \);
- *binary* if every \( R \in Q \) is binary.

Our objective is to design algorithms for answering simple binary queries.

The integer

\[
m = \sum_{R \in Q} |R|
\]

is the *input size* of \( Q \). Concentrating on *data complexity*, we will assume that both \( |Q| \) and \( |\text{attset}(Q)| \) are constants.

**Computation Model.** We will assume the *massively parallel computation* (MPC) model, which is a widely-accepted abstraction of today’s massively parallel systems. Denote by \( p \) the number of machines. In the beginning, the input elements are evenly distributed across these machines. For a join query, this means that each machine stores \( \Theta(m/p) \) tuples from the input relations (we consider that every value in \( \text{dom} \) can be encoded in a single word).

An algorithm is executed in *rounds*, each having two phases:
- In the first phase, every machine performs computation on the data of its local storage.
- In the second phase, the machines communicate by sending messages to each other.

All the messages sent out in the second phase must be prepared in the first phase. This prevents a machine from, for example, sending information based on the data received during the second phase. Another round is launched only if the current round has not solved the problem. In our context, *solving* a join query means producing every tuple in the join result on at least one machine.

The *load* of a round is the largest number of words received by a machine in this round, that is, if machine \( i \in [1, p] \) receives \( x_i \) words, the load is \( \max_{i=1}^{p} x_i \). The performance of an algorithm is measured by two metrics: (i) the number of rounds, and (ii) the *load* of the algorithm, defined as the total load of all rounds. CPU computation is for free. We will be...
interested only in algorithms finishing in a constant number of rounds. The load of such an algorithm is asymptotically the same as the maximum load of the individual rounds.

The number \( p \) of machines is assumed to be significantly less than \( m \), which in this paper means \( p^3 \leq m \). For a randomized algorithm, when we say that its load is at most \( L \), we mean that its load is bounded by \( L \) with probability at least \( 1 - 1/p^c \) where \( c \) can be set to an arbitrarily large constant. The notation \( \tilde{O}(.) \) hides a factor that is polylogarithmic to \( m \) and \( p \).

1.2. Previous Results. Several algorithms, collectively known as the hypercube method, were described in [3] and [6] for parallel join evaluation, but the analysis was restricted to data with no skew; those algorithms can incur expensive cost on general inputs. Koutris, Beame, and Suciu [13] achieved a considerably stronger guarantee for arbitrary data inputs by combining the hypercube method with new skew-handling techniques. Their algorithm solves a join query in one round with load \( \tilde{O}(m/p^{1/\rho}) \) where \( \psi \) is the query’s fractional quasi-packing number. They also proved that \( \Omega(m/p^{1/\rho}) \) is a lower bound on the load of any one-round algorithm, under certain restrictions on the statistics available to the algorithm.

For any algorithm that performs only a constant number of rounds, \( \Omega(m/p^{1/\rho}) \) is a lower bound [13] of the achievable load, where \( \rho \) is the fractional edge covering number of the query (Section 2 will explain \( \rho \) in detail). In general, the value of \( \rho \) never exceeds, but can be strictly smaller than, \( \psi \), which implies that multi-round algorithms may require significantly smaller loads than the one-round counterparts. Matching the lower bound \( \Omega(m/p^{1/\rho}) \) with a concrete algorithm has been shown possible for several special query classes, including star joins [3], cycle joins [13], clique joins [13], line joins [3,13], Loomis-Whitney joins [13], etc. The simple binary join defined in Section 1.1 captures cycle, clique, and line joins as special cases. Guaranteeing a load of \( O(m/p^{1/\rho}) \) for arbitrary simple binary queries is still open.

1.3. Our Contributions. The paper’s main algorithmic contribution is to settle any simple binary join \( Q \) under the MPC model with load \( \tilde{O}(m/p^{1/\rho}) \) in a constant number rounds (Theorem 13). The load is optimal up to a polylogarithmic factor. Our algorithm owes to a new theorem — we name the isolated cartesian product theorem (Theorem 7; see also Theorem 10) — that reveals a non-trivial fact on the problem’s mathematical structure.

Overview of Our Techniques. Consider the join query \( Q \) illustrated by the graph in Figure 1a. An edge connecting vertices \( X \) and \( Y \) represents a relation \( R_{\{X,Y\}} \) with scheme \( \{X,Y\} \). \( Q \) contains all the 18 relations represented by the edges in Figure 1a; \( \text{attset}(Q) = \{A, B, ..., L\} \) has a size of 12.

Set \( \lambda = \Theta(p^{1/(2p)}) \) where \( \rho \) is the fractional edge covering number of \( Q \) (Section 2). A value \( x \in \text{dom} \) is heavy if at least \( m/\lambda \) tuples in an input relation \( R \in Q \) carry \( x \) on the same attribute. The number of heavy values is \( O(\lambda) \). A value \( x \in \text{dom} \) is light if \( x \) appears in at least one relation \( R \in Q \) but is not heavy. A tuple in the join result may take a heavy or light value on each of the 12 attributes \( A, ..., K \). As there are \( O(\lambda) \) choices on each attribute (i.e., either a light value or one of the \( O(\lambda) \) heavy values), there are \( t = O(\lambda^{12}) \) “choice combinations” from all attributes; we will refer to each combination as a configuration. Our plan is to partition the set of \( p \) servers into \( t \) subsets of sizes \( p_1, p_2, ..., p_t \) with \( \sum_{i=1}^{t} p_i = p \), and then dedicate the \( p_i \) servers (\( 1 \leq i \leq t \)) to computing the result tuples of the \( i \)-th configuration. This can be done in parallel for all \( O(\lambda^{12}) \) configurations. The challenge is to
Accordingly, vertices $D$ we proceed in the way explained next).

where every tuple is “piggybacked” the value $d$ on servers are available for this subtask.

Perform a semi-join reduction.

(1) For each edge $\{X, Y\}$ with two white vertices, $Q'$ has a relation $R_{\{X,Y\}}'$ that contains only the tuples in $R_{\{X,Y\}} \in Q$ using light values on both $X$ and $Y$;

For each edge $\{X, Y\}$ with a white vertex $X$ and a black vertex $Y$, $Q'$ has a relation $R_{\{X,Y\}}'$ that contains only the tuples in $R_{\{X,Y\}} \in Q$ each using a light value on $X$ and the constrained heavy value on $Y$;

For each edge $\{X, Y\}$ with two black vertices, $Q'$ has a relation $R_{\{X,Y\}}'$ with only one tuple that takes the constrained heavy values on $X$ and $Y$, respectively.

For example, a tuple in $R_{\{A,B\}}'$ must use light values on both $A$ and $B$; a tuple in $R_{\{D,G\}}'$ must use value $d$ on $D$ and a light value on $G$; $R_{\{D,K\}}'$ has only a single tuple with values $d$ and $k$ on $D$ and $K$, respectively. Finding all result tuples for $Q$ under the designated configuration amounts to evaluating $Q'$.

Since the black attributes have had their values fixed in the configuration, they can be deleted from the residual query, after which some relations in $Q'$ become unary or even disappear. Relation $R_{\{A,D\}}' \in Q'$, for example, can be regarded as a unary relation over $\{A\}$ where every tuple is “piggybacked” the value $d$ on $D$. Let us denote this unary relation as $R_{\{A\}|d}'$, which is illustrated in Figure 1c with a dotted edge extending from $A$ and carrying the label $d$. The deletion of $D, E, F$, and $K$ results in 13 unary relations (e.g., two of them are over $\{A\}$: $R_{\{A\}|d}'$ and $R_{\{A\}|e}'$). Attributes $G, H, L$ become isolated because they are not connected to any other vertices by solid edges. Relations $R_{\{A,B\}}', R_{\{A,C\}}', R_{\{B,C\}}',$ and $R_{\{L,J\}}'$ remain binary, whereas $R_{\{B,K\}}'$ has disappeared (more precisely, if $R_{\{D,K\}}$ does not contain a tuple taking values $d$ and $k$ on $D$ and $K$, respectively, then $Q'$ has an empty answer; otherwise, we proceed in the way explained next).

Our algorithm solves the residual query $Q'$ of Figure 1c as follows:

(1) Perform a semi-join reduction. There are two steps. First, for every vertex $X$ in Figure 1c, intersect all the unary relations over $\{X\}$ (if any) into a single list $R_{\{X\}}''$. For example, the two unary relations $R_{\{A\}|d}'$ and $R_{\{A\}|e}'$ of $A$ are intersected to produce $R_{\{A\}}''$; only the values in the intersection can appear in the join result. Second, for every
non-isolated attribute $X$ in Figure 1c, use $R''_{\{X\}}$ to shrink each binary relation $R'_{\{X,Y\}}$ (for all relevant $Y$) to eliminate tuples whose $X$-values are absent in $R''_{\{X\}}$. This reduces $R'_{\{X,Y\}}$ to a subset $R''_{\{X,Y\}}$. For example, every tuple in $R''_{\{A,B\}}$ uses an $A$-value from $R''_{\{A\}}$ and a $B$-value from $R''_{\{B\}}$.

(2) **Compute a cartesian product.** The residual query $Q'$ can now be further simplified into a join query $Q''$ which includes (i) the relation $R''_{\{X\}}$ for every isolated attribute $X$, and (ii) the relation $R''_{\{X,Y\}}$ for every solid edge in Figure 1c. Figure 1d gives a neater view of $Q''$; clearly, $\text{Join}(Q'')$ is the cartesian product of $R''_{\{G\}}, R''_{\{H\}}, R''_{\{L\}}, R''_{\{I,J\}}$, and the result of the “triangle join" \{\(R''_{\{A,B\}}, R''_{\{A,C\}}, R''_{\{B,C\}}\).

As mentioned earlier, we plan to use only a small subset of the $p$ servers to compute $Q'$. It turns out that the load of our strategy depends heavily on the cartesian product of the unary relations $R''_{\{X\}}$ (one for every isolated attribute $X$, i.e., $R''_{\{G\}}, R''_{\{H\}}$, and $R''_{\{L\}}$ in our example) in a configuration. Ideally, if the cartesian product of every configuration is small, we can prove a load of $\tilde{O}(m/p^{1/\rho})$ easily. Unfortunately, this is not true: in the worst case, the cartesian products of various configurations can differ dramatically.

Our isolated cartesian product theorem (Theorem 7) shows that the cartesian product size is small when averaged over all the possible configurations. This property allows us to allocate a different number of machines to process each configuration in parallel while ensuring that the total number of machines required will not exceed $p$. The theorem is of independent interest and may be useful for developing join algorithms under other computation models (e.g., the external memory model [4]; see Section 7).

### 1.4. An Application: Subgraph Enumeration.

The joins studied in this paper bear close relevance to the subgraph enumeration problem, where the goal is to find all occurrences of a pattern subgraph $G' = (V', E')$ in a graph $G = (V, E)$. This problem is NP-hard [7] if the size of $G'$ is unconstrained, but is polynomial-time solvable when $G'$ has only a constant number of vertices. In the MPC model, the edges of $G$ are evenly distributed onto the $p$ machines at the beginning, whereas an algorithm must produce every occurrence on at least one machine in the end. The following facts are folklore regarding a constant-size $G'$:

- Every constant-round subgraph enumeration algorithm must incur a load of $\Omega(|E|/p^{1/\rho})$,\(^1\) where $\rho$ is the fractional edge covering number (Section 2) of $G'$.
- The subgraph enumeration problem can be converted to a simple binary join with input size $O(|E|)$ and the same fractional edge covering number $\rho$.

Given a constant-size $G'$, our join algorithm (Theorem 13) solves subgraph enumeration with load $\tilde{O}(|E|/p^{1/\rho})$, which is optimal up to a polylogarithmic factor.

### 1.5. Remarks.

This paper is an extension of both [12] and [19]. Ketsman and Suciu [12] were the first to discover a constant-round algorithm to solve simple binary joins with an asymptotically optimal load. Tao [19] introduced a preliminary version of the isolated cartesian product theorem and applied it to simplify the algorithm of [12]. The current work features a more powerful version of the isolated cartesian product theorem. Table 1 lists the symbols that will be frequently used.

\(^1\)Here, we consider $|E| \geq |V|$ because vertices with no edges can be discarded directly.
2. Hypergraphs and the AGM Bound

We define a hypergraph $G$ as a pair $(V, E)$ where:

- $V$ is a finite set, where each element is called a vertex;
- $E$ is a set of non-empty subsets of $V$, where each subset is called a (hyper-)edge.

Given a vertex $X \in V$ and an edge $e \in E$, we say that $X$ and $e$ are incident to each other if $X \in e$. All hypergraphs discussed in this paper have the property that every vertex is incident to at least one edge.

Two distinct vertices $X, Y \in V$ are adjacent if there is an $e \in E$ containing $X$ and $Y$. An edge $e$ is unary or binary if $|e| = 1$ or 2, respectively. A hypergraph with only binary edges is binary. Given a subset $V'$ of $V$, we define the subgraph induced by $V'$ as $(V', E')$ where $E' = \{X \cap e \mid e \in E \land V' \cap e \neq \emptyset\}$.

<table>
<thead>
<tr>
<th>symbol</th>
<th>meaning</th>
<th>definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$</td>
<td>number of machines</td>
<td>Sec 1.1</td>
</tr>
<tr>
<td>$Q$</td>
<td>join query</td>
<td>Sec 1.1</td>
</tr>
<tr>
<td>$m$</td>
<td>input size of $Q$</td>
<td>(1.1)</td>
</tr>
<tr>
<td>$\text{Join}(Q)$</td>
<td>result of $Q$</td>
<td>Sec 1.1</td>
</tr>
<tr>
<td>$\text{attset}(Q)$</td>
<td>set of attributes in the relations of $Q$</td>
<td>Sec 1.1</td>
</tr>
<tr>
<td>$G(V, E)$</td>
<td>hypergraph of $Q$</td>
<td>Sec 2</td>
</tr>
<tr>
<td>$W$</td>
<td>fractional edge covering/packing of $G$</td>
<td>Sec 2</td>
</tr>
<tr>
<td>$W(e)$</td>
<td>Weight of an edge $e \in E$</td>
<td>Sec 2</td>
</tr>
<tr>
<td>$\rho$ (or $\tau$)</td>
<td>fractional edge covering (or packing) number of $G$</td>
<td>Sec 2</td>
</tr>
<tr>
<td>$R_e(e \in E)$</td>
<td>relation $R \in Q$ with scheme($R$) = $e$</td>
<td>Sec 2</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>heavy parameter</td>
<td>Sec 4</td>
</tr>
<tr>
<td>$\mathcal{H}$</td>
<td>set of heavy attributes in $\text{attset}(Q)$</td>
<td>Sec 4</td>
</tr>
<tr>
<td>$\text{config}(Q, \mathcal{H})$</td>
<td>set of configurations of $\mathcal{H}$</td>
<td>Sec 4</td>
</tr>
<tr>
<td>$\eta$</td>
<td>configuration</td>
<td>Sec 4</td>
</tr>
<tr>
<td>$R'_e(\eta)$</td>
<td>residual relation of $e \in E$ under $\eta$</td>
<td>Sec 4</td>
</tr>
<tr>
<td>$Q'(\eta)$</td>
<td>residual query under $\eta$</td>
<td>(4.1)</td>
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<tr>
<td>$k$</td>
<td>size of $\text{attset}(Q)$</td>
<td>Lemma 6</td>
</tr>
<tr>
<td>$m_{\eta}$</td>
<td>input size of $Q'(\eta)$</td>
<td>Lemma 6</td>
</tr>
<tr>
<td>$\mathcal{L}$</td>
<td>set of light attributes in $\text{attset}(Q)$</td>
<td>(5.2)</td>
</tr>
<tr>
<td>$\mathcal{I}$</td>
<td>set of isolated attributes in $\text{attset}(Q)$</td>
<td>(5.3)</td>
</tr>
<tr>
<td>$R'_X(\eta)$</td>
<td>relation on attribute $X$ after semi-join reduction</td>
<td>(5.4)</td>
</tr>
<tr>
<td>$R'_e(\eta)$</td>
<td>relation on $e \in E$ after semi-join reduction</td>
<td>Sec 5.2</td>
</tr>
<tr>
<td>$Q''_{\text{isolated}}(\eta)$</td>
<td>query on the isolated attributes after semi-join reduction</td>
<td>(5.5)</td>
</tr>
<tr>
<td>$Q''_{\text{light}}(\eta)$</td>
<td>query on the light edges after semi-join reduction</td>
<td>(5.6)</td>
</tr>
<tr>
<td>$Q''(\eta)$</td>
<td>reduced query under $\eta$</td>
<td>(5.7)</td>
</tr>
<tr>
<td>$W_I$</td>
<td>total weight of all vertices in $\mathcal{I}$ under fractional edge packing $W$</td>
<td>(5.10)</td>
</tr>
<tr>
<td>$\mathcal{J}$</td>
<td>non-empty subset of $\mathcal{I}$</td>
<td>Sec 5.4</td>
</tr>
<tr>
<td>$Q''_{\mathcal{J}}(\eta)$</td>
<td>query on the isolated attributes in $\mathcal{J}$ after semi-join reduction</td>
<td>(5.14)</td>
</tr>
<tr>
<td>$W_{\mathcal{J}}$</td>
<td>total weight of all vertices in $\mathcal{J}$ under fractional edge packing $W$</td>
<td>(5.15)</td>
</tr>
</tbody>
</table>

Table 1: Frequently used notations
Fractional Edge Coverings and Packings. Let \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \) be a hypergraph and \( W \) be a function mapping \( \mathcal{E} \) to real values in \([0, 1]\). We call \( W(e) \) the weight of edge \( e \) and \( \sum_{e \in \mathcal{E}} W(e) \) the total weight of \( W \). Given a vertex \( X \in \mathcal{V} \), we refer to \( \sum_{e \in \mathcal{E}, X \in e} W(e) \) (i.e., the sum of the weights of all the edges incident to \( X \)) as the weight of \( X \).

\( W \) is a fractional edge covering of \( \mathcal{G} \) if the weight of every vertex \( X \in \mathcal{V} \) is at least 1. The fractional edge covering number of \( \mathcal{G} \), which is denoted as \( \rho(\mathcal{G}) \), equals the smallest total weight of all the fractional edge coverings. \( W \) is a fractional edge packing if the weight of every vertex \( X \in \mathcal{V} \) is at most 1. The fractional edge packing number of \( \mathcal{G} \), which is denoted as \( \tau(\mathcal{G}) \), equals the largest total weight of all the fractional edge packings. A fractional edge packing \( W \) is tight if it is simultaneously also a fractional edge covering; likewise, a fractional edge covering \( W \) is tight if it is simultaneously also a fractional edge packing. Note that in a tight fractional edge covering/packing, the weight of every vertex must be exactly 1.

Binary hypergraphs have several interesting properties:

Lemma 1. If \( \mathcal{G} \) is binary, then:

- \( \rho(\mathcal{G}) + \tau(\mathcal{G}) = |\mathcal{V}| \); furthermore, \( \rho(\mathcal{G}) \geq \tau(\mathcal{G}) \), where the equality holds if and only if \( \mathcal{G} \) admits a tight fractional edge packing (a.k.a. tight fractional edge covering).
- \( \mathcal{G} \) admits a fractional edge packing \( W \) of total weight \( \tau(\mathcal{G}) \) such that
  1. the weight of every vertex \( X \in \mathcal{V} \) is either 0 or 1;
  2. if \( Z \) is the set of vertices in \( \mathcal{V} \) with weight 0, then \( \rho(\mathcal{G}) - \tau(\mathcal{G}) = |Z| \).

Proof. The first bullet is proved in Theorem 2.2.7 of [18]. The fractional edge packing \( W \) in Theorem 2.1.5 of [18] satisfies Property (1) of the second bullet. Regarding such a \( W \), we have

\[
\tau(\mathcal{G}) = \text{total weight of } W = \frac{1}{2} \sum_{X \in \mathcal{V}} \text{(weight of } X) = (|\mathcal{V}| - |Z|)/2.
\]

Plugging this into \( \rho(\mathcal{G}) + \tau(\mathcal{G}) = |\mathcal{V}| \) yields \( \rho(\mathcal{G}) = (|\mathcal{V}| + |Z|)/2 \). Hence, Property (2) holds.

Example. Suppose that \( \mathcal{G} \) is the binary hypergraph in Figure 1a. It has a fractional edge covering number \( \rho(\mathcal{G}) = 6.5 \), as is achieved by the function \( W_1 \) that maps \{G, F\}, \{D, K\}, \{I, J\}, \{E, H\}, and \{E, L\} to 1, \{A, B\}, \{A, C\}, and \{B, C\} to 1/2, and the other edges to 0. Its fractional edge packing number is \( \tau(\mathcal{G}) = 5.5 \), achieved by the function \( W_2 \) which is the same as \( W_1 \) except that \( W_2 \) maps \{E, L\} to 0. Note that \( W_2 \) satisfies both properties of the second bullet (here \( Z = \{L\} \)).

Hypergraph of a Join Query and the AGM Bound. Every join \( Q \) defines a hypergraph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \) where \( \mathcal{V} = \text{attset}(Q) \) and \( \mathcal{E} = \{\text{scheme}(R) \mid R \in Q\} \). When \( Q \) is simple, for each edge \( e \in \mathcal{E} \) we denote by \( R_e \) the input relation \( R \in Q \) with \( e = \text{scheme}(R) \). The following result is known as the AGM bound:

Lemma 2 [5]. Let \( Q \) be a simple binary join query, and \( W \) be any fractional edge covering of the hypergraph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \) defined by \( Q \). Then, \(|\text{Join}(Q)| \leq \prod_{e \in \mathcal{E}} |R_e|^{|W(e)|}\).
3. **Fundamental MPC Algorithms**

This subsection will discuss several building-block routines in the MPC model useful later.

**Cartesian Products.** Suppose that $R$ and $S$ are relations with disjoint schemes. Their cartesian product, denoted as $R \times S$, is a relation over $\text{scheme}(R) \cup \text{scheme}(S)$ that consists of all the tuples $u$ over $\text{scheme}(R) \cup \text{scheme}(S)$ such that $u[\text{scheme}(R)] \in R$ and $u[\text{scheme}(S)] \in S$.

The lemma below gives a deterministic algorithm for computing the cartesian product:

**Lemma 3.** Let $Q$ be a set of $t = \Theta(1)$ relations $R_1, R_2, ..., R_t$ with disjoint schemes. The tuples in $R_i$ ($1 \leq i \leq t$) are labeled with 1, 2, ..., $|R_i|$, respectively. We can deterministically compute $\text{Join}(Q) = R_1 \times R_2 \times ... \times R_t$ in one round with load

$$O\left(\frac{\max_{\text{non-empty } Q' \subseteq Q} |\text{Join}(Q')|}{p^{\frac{1}{|Q'|}}} \right)$$

using $p$ machines. Alternatively, if we assume $|R_1| \geq |R_2| \geq ... \geq |R_t|$, then the load can be written as

$$O\left(\sum_{i=1}^{t} \frac{|\text{Join}([R_1, R_2, ..., R_i])|^{\frac{1}{2}}}{p^i} \right).$$

**Proof.** For each $i \in [1, t]$, define $Q_i = \{R_1, ..., R_i\}$ and $L_i = |\text{Join}(Q_i)|^{\frac{1}{2}}/p^i$. Let $t'$ be the largest integer satisfying $|R_i| \geq L_i$ for all $i \in [1, t']$; $t'$ definitely exists because $|R_1| \geq L_1 = |R_1|/p$. Note that this means $|R_i| \leq |R_{t-1}| \leq ... \leq |R_{t'+1}| < L_{t'+1}$ if $t' < t$.

Next, we will explain how to obtain $\text{Join}(Q_{t'})$ with load $O(L_{t'})$. If $t' < t$, this implies that $\text{Join}(Q)$ can be obtained with load $O(L_{t'} + L_{t'+1})$ because $R_{t'+1}, ..., R_t$ can be broadcast to all the machines with an extra load $O(L_{t'+1} \cdot (t - t')) = O(L_{t'+1})$.

Align the machines into a $t'$-dimensional $p_1 \times p_2 \times ... \times p_{t'}$ grid where $p_i = \lceil |R_i|/L_{t'} \rceil$ for each $i \in [1, t']$. This is possible because $|R_i| \geq |R_{t'}| \geq L_{t'}$ and $\prod_{i=1}^{t'} \frac{|R_i|}{L_{t'}} = \frac{|\text{Join}(Q_{t'})|}{(L_{t'})^{t'}} = p$.

Each machine can be uniquely identified as a $t'$-dimensional point $(x_1, ..., x_{t'})$ in the grid where $x_i \in [1, p_i]$ for each $i \in [1, t']$. For each $R_i$, we send its tuple with label $j \in [1, |R_i|]$ to all the machines whose coordinates on dimension $i$ are $(j \mod p_i) + 1$. Hence, a machine receives $O(|R_i|/p_i) = O(L_{t'})$ tuples from $R_i$; and the overall load is $O(L_{t'})$. For each combination of $u_1, u_2, ..., u_{t'}$ where $u_i \in R_i$, some machine has received all of $u_1, ..., u_{t'}$. Therefore, the algorithm is able to produce the entire $\text{Join}(Q_{t'})$.

The load in (3.2) matches a lower bound stated in Section 4.1.5 of [14]. The algorithm in the above proof generalizes an algorithm in [10] for computing the cartesian product of $t = 2$ relations. The randomized hypercube algorithm of [6] incurs a load higher than (3.2) by a logarithmic factor and can fail with a small probability.

**Composition by Cartesian Product.** If we already know how to solve queries $Q_1$ and $Q_2$ separately, we can compute the cartesian product of their results efficiently:
Lemma 4. Let $Q_1$ and $Q_2$ be two join queries satisfying the condition $\text{attset}(Q_1) \cap \text{attset}(Q_2) = \emptyset$. Let $m$ be the total number of tuples in the input relations of $Q_1$ and $Q_2$. Suppose that

- with probability at least $1 - \delta_1$, we can compute in one round $\text{Join}(Q_1)$ with load $\tilde{O}(m/p_1^{1/t_1})$ using $p_1$ machines;
- with probability at least $1 - \delta_2$, we can compute in one round $\text{Join}(Q_2)$ with load $\tilde{O}(m/p_2^{1/t_2})$ using $p_2$ machines.

Then, with probability at least $1 - \delta_1 - \delta_2$, we can compute $\text{Join}(Q_1) \times \text{Join}(Q_2)$ in one round with load $\tilde{O}(\max\{m/p_1^{1/t_1}, m/p_2^{1/t_2}\})$ using $p_1p_2$ machines.

Proof. Let $A_1$ and $A_2$ be the algorithm for $Q_1$ and $Q_2$, respectively. If a tuple $u \in \text{Join}(Q_1)$ is produced by $A_1$ on the $i$-th $(i \in [1, p_1])$ machine, we call $u$ an $i$-tuple. Similarly, if a tuple $v \in \text{Join}(Q_2)$ is produced by $A_2$ on the $j$-th $(j \in [1, p_2])$ machine, we call $v$ a $j$-tuple.

Arrange the $p_1p_2$ machines into a matrix where each row has $p_1$ machines and each column has $p_2$ machines (note that the number of rows is $p_2$ while the number of columns is $p_1$). For each row, we run $A_1$ using the $p_1$ machines on that row to compute $\text{Join}(Q_1)$; this creates $p_2$ instances of $A_1$ (one per row). If $A_1$ is randomized, we instruct all those instances to take the same random choices. This ensures:

- with probability $1 - \delta_1$, all the instances succeed simultaneously;
- for each $i \in [1, p_1]$, all the machines at the $i$-th column produce exactly the same set of $i$-tuples.

The load incurred is $\tilde{O}(m/p_1^{1/t_1})$. Likewise, for each column, we run $A_2$ using the $p_2$ machines on that column to compute $\text{Join}(Q_2)$. With probability $1 - \delta_2$, for each $j \in [1, p_2]$, all the machines at the $j$-th row produce exactly the same set of $j$-tuples. The load is $\tilde{O}(m/p_2^{1/t_2})$.

Therefore, it holds with probability at least $1 - \delta_1 - \delta_2$ that, for each pair $(i, j)$, some machine has produced all the $i$- and $j$-tuples. Hence, every tuple of $\text{Join}(Q_1) \times \text{Join}(Q_2)$ appears on a machine. The overall load is the larger between $\tilde{O}(m/p_1^{1/t_1})$ and $\tilde{O}(m/p_2^{1/t_2})$.

**Skew-Free Queries.** It is possible to solve a join query $Q$ on binary relations in a single round with a small load if no value appears too often. To explain, denote by $m$ the input size of $Q$; set $k = |\text{attset}(Q)|$, and list out the attributes in $\text{attset}(Q)$ as $X_1, \ldots, X_k$. For $i \in [1, k]$, let $p_i$ be a positive integer referred to as the share of $X_i$. A relation $R \in Q$ with scheme $\{X_i, X_j\}$ is skew-free if every value $x \in \text{dom}$ fulfills both conditions below:

- $R$ has $O(m/p_i)$ tuples $u$ with $u(X_i) = x$;
- $R$ has $O(m/p_j)$ tuples $u$ with $u(X_j) = x$.

Define $\text{share}(R) = p_i \cdot p_j$. If every $R \in Q$ is skew-free, $Q$ is skew-free. We know:

**Lemma 5 [6].** With probability at least $1 - 1/p^c$ where $p = \prod_{i=1}^{k} p_i$ and $c \geq 1$ can be set to an arbitrarily large constant, a skew-free query $Q$ can be answered in one round with load $\tilde{O}(m/\min_{R \in Q} \text{share}(R))$ using $p$ machines.
4. A Taxonomy of the Join Result

Given a simple binary join \( Q \), we will present a method to partition \( \text{Join}(Q) \) based on the value frequencies in the relations of \( Q \). Denote by \( G = (\mathcal{V}, \mathcal{E}) \) the hypergraph defined by \( Q \) and by \( m \) the input size of \( Q \).

**Heavy and Light Values.** Fix an arbitrary integer \( \lambda \in [1, m] \). A value \( x \in \text{dom} \) is
- **heavy** if \(|\{u \in R \mid u(X) = x\}| \geq m/\lambda \) for some relation \( R \in Q \) and some attribute \( X \in \text{scheme}(R) \);
- **light** if \( x \) is not heavy, but appears in at least one relation \( R \in Q \).

Because each relation has \( O(1) \) attributes, the number of heavy values is \( O(\lambda) \). We will refer to \( \lambda \) as the **heavy parameter**.

**Configurations.** Let \( \mathcal{H} \) be an arbitrary (possibly empty) subset of \( \text{attset}(Q) \). A **configuration** of \( \mathcal{H} \) is a tuple \( \eta \) over \( \mathcal{H} \) such that \( \eta(X) \) is heavy for every \( X \in \mathcal{H} \). Let \( \text{config}(Q, \mathcal{H}) \) be the set of all configurations of \( \mathcal{H} \). It is clear that \( |\text{config}(Q, \mathcal{H})| = O(\lambda^{|\mathcal{H}|}) \).

**Residual Relations/Queries.** Consider an edge \( e \in \mathcal{E} \); define \( e' = e \setminus \mathcal{H} \). We say that \( e \) is **active** on \( \mathcal{H} \) if \( e' \neq \emptyset \), i.e., \( e \) has at least one attribute outside \( \mathcal{H} \). An active \( e \) defines a **residual relation** under \( \eta \) — denoted as \( R'_e(\eta) \) — which
- is over \( e' \) and
- consists of every tuple \( v \) that is the projection (on \( e' \)) of some tuple \( w \in R_e \) “consistent” with \( \eta \), namely:
  - \( w(X) = \eta(X) \) for every \( X \in e \cap \mathcal{H} \);
  - \( w(Y) \) is light for every \( Y \in e' \);
  - \( v = w[e'] \).

The **residual query** under \( \eta \) is
\[
Q'(\eta) = \{ R'_e(\eta) \mid e \in \mathcal{E}, \ e \text{ active on } \mathcal{H} \}.
\] (4.1)

**Example.** Consider the query \( Q \) in Section 1.3 (hypergraph \( G \) in Figure 1a) and the configuration \( \eta \) of \( \mathcal{H} = \{D, E, F, K\} \) where \( \eta[D] = d, \eta[E] = e, \eta[F] = f, \eta[K] = k \). If \( e \) is the edge \( \{A, D\} \), then \( e' = \{A\} \) and \( R'_e(\eta) \) is the relation \( R'_{\{A\},d} \) mentioned in Section 1.3. If \( e \) is the edge \( \{A, B\} \), then \( e' = \{A, B\} \) and \( R'_e(\eta) \) is the relation \( R'_{\{A,B\}} \) in Section 1.3. The residual query \( Q'(\eta) \) is precisely the query \( Q' \) in Section 1.3.

It is rudimentary to verify
\[
\text{Join}(Q) = \bigcup_{\mathcal{H}} \left( \bigcup_{\eta \in \text{config}(Q, \mathcal{H})} \text{Join}(Q'(\eta)) \times \{\eta\} \right). \] (4.2)

**Lemma 6.** Let \( Q \) be a simple binary join and \( \mathcal{H} \) be a subset of \( \text{attset}(Q) \). For each configuration \( \eta \in \text{config}(Q, \mathcal{H}) \), denote by \( m_\eta \) the total size of all the relations in \( Q'(\eta) \). We have:
\[
\sum_{\eta \in \text{config}(Q, \mathcal{H})} m_\eta = O(m \cdot \lambda^{k-2})
\]
where \( k = |\text{attset}(Q)| \).
Proof. Let \( e \) be an edge in \( E \). A tuple \( u \in R_e \) contributes 1 to the term \( m_\eta \) only if \( \eta(X) = u(X) \) for every attribute \( X \in e \cap H \). There are \( O(\lambda^{|H \setminus e|}) \) such \( \eta \). Observe that \( |H \setminus e| \) is at most \( k - 2 \) because \( |H| \leq k \) and \( e \) has two attributes. The lemma thus follows. \( \square \)

5. A Join Computation Framework

Answering a simple binary join \( Q \) amounts to producing the right-hand side of (4.2). Due to symmetry, it suffices to explain how to do so for an arbitrary subset \( H \subseteq \text{attset}(Q) \), i.e., the computation of

\[
\bigcup_{\eta \in \text{config}(Q,H)} \text{Join}(Q'(\eta)).
\]  

(5.1)

At a high level, our strategy (illustrated in Section 1.3) works as follows. Let \( G = (V,E) \) be the hypergraph defined by \( Q \). We will remove the vertices in \( H \) from \( G \), which disconnects \( G \) into connected components (CCs). We divide the CCs into two groups: (i) the set of CCs each involving at least 2 vertices, and (ii) the set of all other CCs, namely those containing only 1 vertex. We will process the CCs in Group 1 \textit{together} using Lemma 5, process the CCs in Group 2 \textit{together} using Lemma 3, and then compute the cartesian product between Groups 1 and 2 using Lemma 4.

Sections 5.1 and 5.2 will formalize the strategy into a processing framework. Sections 5.3 and 5.4 will then establish two important properties of this framework, which are the key to its efficient implementation in Section 6.

5.1. Removing the Attributes in \( H \). We will refer to each attribute in \( H \) as a \textit{heavy attribute}. Define

\[
\mathcal{L} = \text{attset}(Q) \setminus H;
\]  

(5.2)

and call each attribute in \( \mathcal{L} \) a \textit{light attribute}. An edge \( e \in E \) is

- a \textit{light edge} if \( e \) contains two light attributes, or
- a \textit{cross edge} if \( e \) contains a heavy attribute and a light attribute.

A light attribute \( X \in \mathcal{L} \) is a \textit{border attribute} if it appears in at least one cross edge \( e \) of \( G \). Denote by \( G' = (\mathcal{L},E') \) the subgraph of \( G \) induced by \( \mathcal{L} \). A vertex \( X \in \mathcal{L} \) is \textit{isolated} if \( \{X\} \) is the only edge in \( E' \) incident to \( X \). Define

\[
\mathcal{I} = \{X \in \mathcal{L} | X \text{ is isolated}\}.
\]  

(5.3)

Example (cont.). Consider again the join query \( Q \) whose hypergraph \( G \) is shown in Figure 1a. Set \( H = \{D,E,F,K\} \). Set \( \mathcal{L} \) includes all the white vertices in Figure 1b. Edge \( \{A,B\} \) is a light edge, \( \{A,D\} \) is a cross edge, while \( \{D,K\} \) is neither a light edge nor a cross edge. All the vertices in \( \mathcal{L} \) except \( J \) are border vertices. Figure 2 shows the subgraph of \( G \) induced by \( \mathcal{L} \), where a unary edge is represented by a box and a binary edge by a segment. The isolated vertices are \( G, H, \text{ and } L \). \( \square \)
5.2. **Semi-Join Reduction.** Recall from Section 4 that every configuration \( \eta \) of \( \mathcal{H} \) defines a residual query \( Q'(\eta) \). Next, we will simplify \( Q'(\eta) \) into a join \( Q''(\eta) \) with the same result.

Observe that the hypergraph defined by \( Q'(\eta) \) is always \( G' = (L, E') \), regardless of \( \eta \). Consider a border attribute \( X \in L \) and a cross edge \( e \) of \( G = (V, E) \) incident to \( X \). As explained in Section 4, the input relation \( R_e \in Q \) defines a unary residual relation \( R'_e(\eta) \in Q'(\eta) \). Note that \( R'_e(\eta) \) has scheme \( \{ X \} \). We define:

\[
R'_X(\eta) = \bigcap_{e \in E \text{ s.t. } X \in e} R'_e(\eta).
\] (5.4)

**Example (cont.).** Let \( \mathcal{H} = \{ D, E, F, K \} \), and consider its configuration \( \eta \) with \( \eta(D) = d, \eta(E) = e, \eta(F) = f, \) and \( \eta(K) = k \). Set \( X \) to the border attribute \( A \). When \( e \in \{ A, D \} \) or \( \{ A, E \} \), \( R'_e(\eta) \) is the relation \( R'_{\{A\}|d} \) or \( R'_{\{A\}|e} \), respectively. \( \{ A, D \} \) and \( \{ A, E \} \) are the only cross edges containing \( A \). Hence, \( R''_A(\eta) = R'_{\{A\}|d} \cap R'_{\{A\}|e} \) which is the relation \( R''_{\{A\}} \) in Section 1.3.

Recall that every light edge \( e = \{ X, Y \} \) in \( G \) defines a residual relation \( R'_e(\eta) \) with scheme \( e \). We define \( R''_e(\eta) \) as a relation over \( e \) that contains every tuple \( u \in R'_e(\eta) \) satisfying:

- (applicable only if \( X \) is a border attribute) \( u(X) \in R'_{X}(\eta) \);
- (applicable only if \( Y \) is a border attribute) \( u(Y) \in R'_{Y}(\eta) \).

Note that if neither \( X \) nor \( Y \) is a border attribute, then \( R''_e(\eta) = R'_e(\eta) \).

**Example (cont.).** For the light edge \( e = \{ A, B \} \), \( R'_e(\eta) \) is the relation \( R'_{\{A,B\}} \) mentioned in Section 1.3. Because \( A \) and \( B \) are border attributes, \( R'_e(\eta) \) includes all the tuples in \( R'_{\{A,B\}} \) that take an \( A \)-value from \( R'_{A}(\eta) \) and a \( B \)-value from \( R'_{B}(\eta) \). This \( R'_e(\eta) \) is precisely the relation \( R''_{\{A,B\}} \) in Section 1.3.

Every vertex \( X \in I \) must be a border attribute and, thus, must now be associated with \( R''_X(\eta) \). We can legally define:

\[
Q''_{\text{isolated}}(\eta) = \{ R''_X(\eta) \mid X \in I \}
\] (5.5)
\[
Q''_{\text{light}}(\eta) = \{ R''_e(\eta) \mid \text{light edge } e \in E \}
\] (5.6)
\[
Q''(\eta) = Q''_{\text{light}}(\eta) \cup Q''_{\text{isolated}}(\eta).
\] (5.7)

**Example (cont.).** \( Q''_{\text{isolated}}(\eta) = \{ R''_{\{G\}}, R''_{\{H\}}, R''_{\{L\}} \} \) and \( Q''_{\text{light}}(\eta) = \{ R''_{\{A,B\}}, R''_{\{A,C\}}, R''_{\{B,C\}}, R_{\{1,J\}} \} \), where all the relation names follow those in Section 1.3.
We will refer to the process of converting \( Q'(\eta) \) to \( Q''(\eta) \) as *semi-join reduction*, and call \( Q''(\eta) \) the *reduced query* under \( \eta \). It is rudimentary to verify:

\[
Join(Q'(\eta)) = Join(Q''(\eta)) = Join(Q''_{isolated}(\eta)) \times Join(Q''_{light}(\eta)).
\]

(5.8)

### 5.3. The Isolated Cartesian Product Theorem

In this subsection, we will focus on analyzing the size of \( \text{Join}(Q''_{isolated}(\eta)) \), which has a crucial impact on the efficiency of our join algorithm. As shown in (5.5), \( Q''_{isolated}(\eta) \) contains \(|I|\) unary relations, one for each isolated attribute in \( I \). In other words, \( \text{Join}(Q''_{isolated}(\eta)) \) is the cartesian product of all those relations. As each relation (trivially) has size at most \( m \), we immediately have

\[
|\text{Join}(Q''_{isolated}(\eta))| \leq m^{|I|}.
\]

This leads to:

\[
\sum_{\eta \in \text{config}(Q,H)} \left| \text{Join}(Q''_{isolated}(\eta)) \right| = O\left(\lambda^{|H|} \cdot m^{|I|}\right)
\]

(5.9)

because \( H \) has \( \lambda^{|H|} \) different configurations. Unfortunately, the above is not enough to prove the target join performance. As shown in Lemma 3, the load for computing a cartesian product depends on the cartesian product’s size; thus, we will have to show that the left-hand side of (5.9) (which is the total size of \( |\text{config}(Q,H)| \) cartesian products) is sufficiently small. Next, we will establish a bound stronger than (5.9).

Take an arbitrary fractional edge packing \( W \) of the hypergraph \( G \). Define

\[
W_T = \sum_{Y \in I} \text{weight of } Y \text{ under } W.
\]

(5.10)

Recall that the weight of a vertex \( Y \) under \( W \) is the sum of \( W(e) \) for all the edges \( e \in E \) containing \( Y \).

**Example.** Let \( G \) be the hypergraph in Figure 1a. Consider the function \( W \) that maps \( \{G,F\}, \{D,K\}, \{I,J\}, \) and \( \{E,H\} \) to 1, \( \{A,B\}, \{A,C\}, \) and \( \{B,C\} \) to 1/2, and the other edges to 0. If \( H = \{D,E,F,K\} \), then \( I = \{G,H,L\} \) and \( W_T = 2 \).

We are ready to present the first main result of this paper:

**Theorem 7 (The isolated cartesian product theorem).** Let \( Q \) be a simple binary query whose relations have a total size of \( m \). Denote by \( G \) the hypergraph defined by \( Q \). Consider an arbitrary subset \( H \subseteq \text{attset}(Q) \), where \( \text{attset}(Q) \) is the set of attributes in the relations of \( Q \). Let \( I \) be the set of isolated vertices defined in (5.3). Take an arbitrary fractional edge packing \( W \) of \( G \). It holds that

\[
\sum_{\eta \in \text{config}(Q,H)} \left| \text{Join}(Q''_{isolated}(\eta)) \right| = O\left(\lambda^{|H| - W_T} \cdot m^{|I|}\right)
\]

(5.11)

where \( \lambda \) is the heavy parameter (Section 4), \( \text{config}(Q,H) \) is the set of configurations of \( H \) (Section 4), \( Q''_{isolated}(\eta) \) is defined in (5.5), and \( W_T \) is defined in (5.10).

Note that (5.11) is tighter than (5.9) whenever \( W_T > 0 \). Theorem 7 is in the strongest form if selecting a \( W \) that maximizes \( W_T \). Later in Section 5.5, we will choose a specific \( W \) that yields a bound sufficient for us to design an efficient join algorithm.

**Proof of Theorem 7.** We will construct a set \( Q^* \) of relations such that \( \text{Join}(Q^*) \) has a result size at least the left-hand side of (5.11). Then, we will prove that the hypergraph of...
$Q^*$ has a fractional edge covering that (by the AGM bound; Lemma 2) implies an upper bound on $|\text{Join}(Q^*)|$ that matches the right-hand side of the equation.

Initially, set $Q^*$ to $\emptyset$. For every cross edge $e \in E$ incident to a vertex in $I$, add to $Q^*$ a relation $R_e = R_e$. For every $X \in H$, add a unary relation $R_{\{X\}}$ to $Q^*$ which consists of all the heavy values on $X$; note that $R_{\{X\}}$ has $O(\lambda)$ tuples. Finally, for every $Y \in I$, add a binary relation $R_{\{Y\}}$ to $Q^*$ which contains all the heavy and light values on $Y$.

Define $G^* = (V^*, E^*)$ as the hypergraph defined by $Q^*$. Note that $V^* = I \cup H$, while $E^*$ consists of all the cross edges in $G$ incident to a vertex in $I$, $|H|$ unary edges $\{X\}$ for every $X \in H$, and $|I|$ unary edges $\{Y\}$ for every $Y \in I$.

**Example (cont.).** Figure 3 shows the hypergraph of the $Q^*$ constructed. As before, a box and a segment represent a unary and a binary edge, respectively. Recall that $H = \{D,E,F,K\}$ and $I = \{G,H,L\}$.

**Lemma 8.** $\sum_{\eta' \in \text{config}(Q, H)} |\text{Join}(Q''_{\text{isolated}}(\eta'))| \leq |\text{Join}(Q^*)|.$

**Proof.** We will prove

$$\bigcup_{\eta' \in \text{config}(Q, H)} \text{Join}(Q''_{\text{isolated}}(\eta')) \times \{\eta'\} \subseteq \text{Join}(Q^*).$$

(5.12)

from which the lemma follows.

Take a tuple $u$ from the left-hand side of (5.12), and set $\eta' = u[H]$. Based on the definition of $Q''_{\text{isolated}}(\eta')$, it is easy to verify that $u[e] \in R_e$ for every cross edge $e \in E$ incident a vertex in $I$; hence, $u[e] \in R^*_e$. Furthermore, $u(X) \in R^*_{\{X\}}$ for every $X \in H$ because $u(X) = \eta'(X)$ is a heavy value. Finally, obviously $u(Y) \in R^*_{\{Y\}}$ for every $Y \in I$. All these facts together ensure that $u \in \text{Join}(Q^*)$. □

**Lemma 9.** $G^*$ admits a tight fractional edge covering $W^*$ satisfying $\sum_{X \in H} W^*(\{X\}) = |H| - W_I$.

**Proof.** We will construct a desired function $W^*$ from the fractional edge packing $W$ in Theorem 7.

For every cross edge $e \in E$ incident to a vertex in $I$, set $W^*(e) = W(e)$. Every edge in $E$ incident to $Y \in I$ must be a cross edge. Hence, $\sum_{\text{binary } e \in E^*: Y \in e} W^*(e)$ is precisely the weight of $Y$ under $W$. 

![Figure 3: Illustration of $Q^*$](image)
Next, we will ensure that each attribute \( Y \in I \) has a weight 1 under \( W^* \). Since \( W \) is a fractional edge packing of \( G \), it must hold that 
\[
\sum_{\text{binary } e \in E^*: Y \in e} W(e) \leq 1.
\]
This permits us to assign the following weight to the unary edge \( \{Y\} \):
\[
W^*(\{Y\}) = 1 - \sum_{\text{binary } e \in E^*: Y \in e} W(e).
\]

Finally, in a similar way, we make sure that each attribute \( X \in H \) has a weight 1 under \( W^* \) by assigning:
\[
W^*(\{X\}) = 1 - \sum_{\text{binary } e \in E^*: X \in e} W(e).
\]
This finishes the design of \( W^* \), which is now a tight fractional edge covering of \( G^* \).

Clearly:
\[
\sum_{X \in H} W^*(\{X\}) = |H| - \sum_{X \in H} \left( \sum_{\text{binary } e \in E^*: X \in e} W(e) \right).
\] (5.13)

Every binary edge \( e \in E^* \) contains a vertex in \( H \) and a vertex in \( I \). Therefore:
\[
\sum_{X \in H} \left( \sum_{\text{binary } e \in E^*: X \in e} W(e) \right) = \sum_{Y \in I} \left( \sum_{\text{binary } e \in E^*: Y \in e} W(e) \right) = W_I.
\]

Putting together the above equation with (5.13) completes the proof.

**Example (cont.).** For the \( G^* \) in Figure 3, a fractional edge covering promised by Lemma 9 is given by the function \( W^* \) that maps \( \{G, F\}, \{E, H\}, \{D\}, \{K\}, \{L\} \) to 1, and the other edges to 0. Note that 
\[
\sum_{X \in H} W^*(\{X\}) = W^*(\{D\}) + W^*(\{K\}) = 2, \text{ same as } |H| - W_I = 4 - 2 = 2.
\]
The AGM bound in Lemma 2 tells us that 
\[
\text{Join}(\mathcal{Q}^*) \leq \prod_{e \in E^*} |R^*_e|^{W^*(e)} = \prod_{X \in H} |R^*_X|^{W^*(\{X\})} \prod_{Y \in I} \prod_{e \in E^*: Y \in e} |R^*_e|^{W^*(e)}
\]  
\[
= \left( \prod_{X \in H} (O(\lambda))^{W^*(\{X\})} \right) \left( \prod_{Y \in I} \prod_{e \in E^*: Y \in e} m^{W^*(e)} \right)
\]  
(by Lemma 9) \[
O(\lambda^{|H| - W_I}) \cdot m^{|I|}
\]
which completes the proof of Theorem 7.

**5.4. A Subset Extension of Theorem 7.** Remember that \( \mathcal{Q}_{\text{isolated}}^*(\eta) \) contains a relation \( R_X^*(\eta) \) (defined in (5.4)) for every attribute \( X \in \mathcal{I} \). Given a non-empty subset \( \mathcal{J} \subseteq \mathcal{I} \), define
\[
\mathcal{Q}_{\mathcal{J}}^*(\eta) = \{R_X^*(\eta) \mid X \in \mathcal{J}\}.
\] (5.14)
Take an arbitrary fractional edge packing \( W \) of the hypergraph \( G \). Define
\[
W_{\mathcal{J}} = \sum_{Y \in \mathcal{J}} \text{weight of } Y \text{ under } W.
\] (5.15)

We now present a general version of the isolated cartesian product theorem:
Theorem 10. Let $Q$ be a simple binary query whose relations have a total size of $m$. Denote by $G$ the hypergraph defined by $Q$. Consider an arbitrary subset $H \subseteq \text{attset}(Q)$, where attset($Q$) is the set of attributes in the relations of $Q$. Let $I$ be the set of isolated vertices defined in (5.3) and $J$ be any non-empty subset of $I$. Take an arbitrary fractional edge packing $W$ of $G$. It holds that

$$
\sum_{\eta \in \text{config}(Q, H)} \left| \text{Join}(Q''(\eta)) \right| = O\left(\lambda^{|H| - W_J} \cdot m^{J} \right).
$$

(5.16)

where $\lambda$ is the heavy parameter (see Section 4), config($Q, H$) is the set of configurations of $H$ (Section 4), $Q''(\eta)$ is defined in (5.14), and $W_J$ is defined in (5.15).

Proof. We will prove the theorem by reducing it to Theorem 7. Define $J = I \setminus J$ and

$$
\hat{Q} = \{ R \in Q \mid \text{scheme}(R) \cap J = \emptyset \}.
$$

In other words, imagine discarding from $Q$ every relation $R$ whose scheme contains an attribute in $J$; then, $Q$ includes the relations remaining in $Q$. Denote by $\hat{G} = (V, \hat{E})$ the hypergraph defined by $\hat{Q}$.

Set $\hat{H} = H \cap \text{attset}(\hat{Q})$ and $\hat{L} = \text{attset}(\hat{Q}) \setminus \hat{H}$. $J$ is precisely the set of isolated attributes decided by $\hat{Q}$ and $\hat{H}$. Define a function $\hat{W} : \hat{E} \to [0, 1]$ by setting $\hat{W}(e) = W(e)$ for every $e \in \hat{E}$. $\hat{W}$ is a fractional edge packing of $\hat{Q}$. Because every edge $e \in \hat{E}$ containing an attribute in $J$ is preserved in $\hat{E}$, we have $W_J = \hat{W}_J$. Applying Theorem 7 to $\hat{Q}$ gives:

$$
\sum_{\hat{\eta} \in \text{config}(\hat{Q}, \hat{H})} \left| \text{Join}(\hat{Q}''(\hat{\eta})) \right| = O\left(\lambda^{|\hat{H}| - |\hat{W}_J|} \cdot m^{J} \right).
$$

(5.17)

It remains to show

$$
\sum_{\eta \in \text{config}(Q, H)} \left| \text{Join}(Q''(\eta)) \right| = O\left(\lambda^{|H| - |\hat{H}|} \sum_{\hat{\eta} \in \text{config}(\hat{Q}, \hat{H})} \left| \text{Join}(\hat{Q}''(\hat{\eta})) \right| \right)
$$

(5.18)

after which Theorem 10 will follow from (5.17) and (5.18).

For each $\eta \in \text{config}(Q, H)$, we can find $\hat{\eta} = \eta | H$ in config($\hat{Q}$, $\hat{H}$) such that $\text{Join}(Q''(\eta)) = \text{Join}(\hat{Q}_{\text{isolated}}''(\eta))$. The correctness of (5.18) follows from the fact that $O(\lambda^{|H| - |\hat{H}|})$ configurations $\eta \in \text{config}(Q, H)$ correspond to the same $\hat{\eta}$. \hfill $\square$

5.5. A Weaker Result. One issue in applying Theorem 10 is that the quantity $|H| - W_J$ is not directly related to the fractional edge covering number $\rho$ of $Q$. The next lemma gives a weaker result that addresses the issue to an extent sufficient for our purposes in Section 6:

Lemma 11. Let $Q$ be a simple binary query whose relations have a total size of $m$. Denote by $G$ the hypergraph defined by $Q$. Consider an arbitrary subset $H \subseteq \text{attset}(Q)$, where attset($Q$) is the set of attributes in the relations of $Q$. Define $L = \text{attset}(Q) \setminus H$ and $I$ as the set of isolated vertices in $L$ (see (5.3)). For any non-empty subset $J \subseteq I$, it holds that

$$
\sum_{\eta \in \text{config}(Q, H)} \left| \text{Join}(Q''(\eta)) \right| = O\left(\lambda^{2\rho - |J|} \cdot m^{J} \right)
$$

(5.19)

where $\rho$ is the fractional edge covering number of $G$, $\lambda$ is the heavy parameter (Section 4), config($Q, H$) is the set of configurations of $H$ (Section 4), and $Q''(\eta)$ is defined in (5.14).
Proof. Let $W$ be an arbitrary fractional edge packing satisfying the second bullet of Lemma 1. The weight of $W$ is the fractional edge packing number $\tau$ of $G$ defined by $Q$. Denote by $Z$ the set of vertices in $G$ whose weights under $W$ are 0. Lemma 1 tells us $\tau + \rho - |Z|$. Set $J_0 = J \cap Z$ and $J_1 = J \setminus J_0$. Because $J_0 \subseteq Z$, we can derive:

$$\tau + |J_0| \leq \rho \Rightarrow |\text{attset}(Q)| - \rho + |J_0| \leq \rho \Rightarrow (|H| + |L|) + (|J| - |J_1|) \leq 2\rho \Rightarrow |H| - |J_1| \leq 2\rho - |J| - |L|.$$

Lemma 11 now follows from Theorem 10 due to $|J_1| = W J$, which holds because every vertex in $J_1$ has a weight 1 under $W$.

6. An MPC Join Algorithm

This section will describe how to answer a simple binary join $Q$ in the MPC model with load $\tilde{O}(m/p_1^{1/\rho})$.

We define a statistical record as a tuple $(R, X, x, cnt)$, where $R$ is a relation in $Q$, $X$ an attribute in scheme$(R)$, $x$ a value in dom, and $cnt$ the number of tuples $u \in R$ with $u(X) = x$. Specially, $(R, \emptyset, \text{nil}, cnt)$ is also a statistical record where $cnt$ gives the number of tuples in $R$ that use only light values. A histogram is defined as the set of statistical records for all possible $R$, $X$, and $x$ satisfying (i) $cnt = \Omega(m/p_1^{1/\rho})$ or (ii) $X = \emptyset$ (and, hence $x = \text{nil}$); note that there are only $O(p_1^{1/\rho})$ such records. We assume that every machine has a local copy of the histogram. By resorting to standard MPC sorting algorithms [9,10], the assumption can be satisfied with a preprocessing that takes constant rounds and load $\tilde{O}(p_1^{1/\rho} + m/p)$.

Henceforth, we will fix the heavy parameter

$$\lambda = \Theta(p_1^{1/(2\rho)})$$

and focus on explaining how to compute (5.1) for an arbitrary subset $H$ of attset$(Q)$. As attset$(Q)$ has $2^k = O(1)$ subsets (where $k = |\text{attset}(Q)|$), processing them all in parallel increases the load by only a constant factor and, as guaranteed by (4.2), discovers the entire Join$(Q)$.

Our algorithm produces (5.1) in three steps:

1. Generate the input relations of the residual query $Q'(\eta)$ of every configuration $\eta$ of $H$ (Section 5.1).
2. Generate the input relations of the reduced query $Q''(\eta)$ of every $\eta$ (Section 5.2).
3. Evaluate $Q''(\eta)$ for every $\eta$.

The number of configurations of $H$ is $O(\lambda^{|H|}) = O(\lambda^k) = O(p_1^{k/(2\rho)})$, which is $O(p)$ because $\rho \geq k/2$ by the first bullet of Lemma 1. Next, we elaborate on the details of each step.

Step 1. Lemma 6 tells us that the input relations of all the residual queries have $O(m \cdot \lambda^{k-2})$ tuples in total. We allocate $p'_{\eta} = \lceil p \cdot \frac{m_{\eta}}{\Theta(m \cdot \lambda^{k-2})} \rceil$ machines to store the relations of $Q'(\eta)$, making sure that $\sum_{\eta} p'_{\eta} \leq p$. Each machine keeps on average

$$O(m_{\eta}/p'_{\eta}) = O(m \cdot \lambda^{k-2}/p) = O(m/p_1^{1/\rho})$$
tuples, where the last equality used \( p \geq k/2 \). Each machine \( i \in [1,p] \) can use the histogram to calculate the input size \( m_\eta \) of \( Q'(\eta) \) precisely for each \( \eta \); it can compute locally the id range of the \( m_\eta \) machines responsible for \( Q'(\eta) \). If a tuple \( u \) in the local storage of machine \( i \) belongs to \( Q'(\eta) \), the machine sends \( u \) to a random machine within that id range. Standard analysis shows that each of the \( m_\eta \) machines receives asymptotically the same number of tuples of \( Q'(\eta) \) (up to an \( O(1) \) factor) with probability at least \( 1 - 1/p^c \) for an arbitrarily large constant \( c \). Hence, Step 1 can be done in a single round with load \( \tilde{O}(m/p^{1/p}) \) with probability at least \( 1 - 1/p^c \).

**Step 2.** Now that all the input relations of each \( Q'(\eta) \) have been stored on \( p'_\eta \) machines, the semi-join reduction in Section 5.2 that converts \( Q'(\eta) \) to \( Q''(\eta) \) is a standard process that can be accomplished \([10]\) with sorting in \( O(1) \) rounds incurring a load of \( \tilde{O}(m_\eta/p'_\eta) = \tilde{O}(m/p^{1/p}) \); see also \([13]\) for a randomized algorithm that performs fewer rounds.

**Step 3.** This step starts by letting each machine know about the value of \( \left| \text{Join}(Q''_{\text{isolated}}(\eta)) \right| \) for every \( \eta \). For this purpose, each machine broadcasts to all other machines how many tuples it has in \( R^*_X(\eta) \) for every \( X \in \mathcal{I} \) and every \( \eta \). Since there are \( O(p) \) different \( \eta \), \( O(p) \) numbers are sent by each machine, such that the load of this round is \( O(p^2) \). From the numbers received, each machine can independently figure out the values of all \( \left| \text{Join}(Q''_{\text{isolated}}(\eta)) \right| \).

We allocate

\[
p''_\eta = \Theta \left( \lambda^{\left| \mathcal{I} \right|} + p \cdot \sum_{\text{non-empty } J \subseteq \mathcal{I}} \left| \text{Join}(Q''_{J}(\eta)) \right| / \lambda^{2p-\left| J \right| - \left| \mathcal{I} \right| \cdot m_{i,J}} \right) \tag{6.1}
\]

machines for computing \( Q'(\eta) \). Notice that

\[
\sum_\eta p''_\eta = \left( \sum_\eta \lambda^{\left| \mathcal{I} \right|} \right) + \left( p \cdot \sum_{\text{non-empty } J \subseteq \mathcal{I}} \sum_\eta \left| \text{Join}(Q''_{J}(\eta)) \right| / \lambda^{2p-\left| J \right| - \left| \mathcal{I} \right| \cdot m_{i,J}} \right) = O(p)
\]

where the equality used Lemma 11, the fact that \( \mathcal{I} \) has constant non-empty subsets, and that \( \sum_\eta \lambda^{\left| \mathcal{I} \right|} \leq \lambda^{\left| \mathcal{I} \right|} \cdot \lambda^{k} \leq \lambda^{k} \leq p \). We can therefore adjust the constants in (6.1) to make sure that the total number of machines needed by all the configurations is at most \( p \).

**Lemma 12.** \( Q''(\eta) \) can be answered in one round with load \( O(m/p^{1/p}) \) using \( p''_\eta \) machines, subject to a failure probability of at most \( 1/p^c \) where \( c \) can be set to an arbitrarily large constant.

**Proof.** As shown in (5.8), \( \text{Join}(Q''(\eta)) \) is the cartesian product of \( \text{Join}(Q''_{\text{isolated}}(\eta)) \) and \( \text{Join}(Q''_{\text{light}}(\eta)) \). We deploy \( \Theta(p''_\eta / \lambda^{\left| \mathcal{I} \right|}) \) machines to deterministically compute \( \text{Join}(Q''_{\text{isolated}}(\eta)) \) in one round. By Lemma 3, the load is

\[
\tilde{O} \left( \left| \text{Join}(Q''_{J}(\eta)) \right|^{1/\left| J \right|} / \left( \frac{p''_\eta}{\lambda^{\left| \mathcal{I} \right|}} \right)^{1/\left| J \right|} \right) \tag{6.2}
\]

for some non-empty \( J \subseteq \mathcal{I} \). (6.1) guarantees that

\[
p''_\eta = \Omega \left( p \cdot \left| \text{Join}(Q''_{J}(\eta)) \right| / \lambda^{2p-\left| J \right| - \left| \mathcal{I} \right| \cdot m_{i,J}} \right)
\]
with which we can derive
\[
(6.2) = \tilde{O} \left( \frac{m \cdot \lambda}{p^{1/|J|}} \right) = \tilde{O} \left( \frac{m \cdot \lambda}{p^{1/|J|}} \right) = \tilde{O} \left( \frac{m \cdot p^{2c-2|J|}}{p^{1/|J|}} \right) = \tilde{O} \left( \frac{m}{p^{1/\rho}} \right).
\]

Regarding \( Q''_{\text{light}}(\eta) \), first note that \( \text{attsset}(Q''_{\text{light}}(\eta)) = L \setminus I \). If \( L \setminus I \) is empty, no \( Q''_{\text{light}}(\eta) \) exists and \( \text{Join}(Q''(\eta)) = \text{Join}(Q''_{\text{isolated}}(\eta)) \). The subsequent discussion considers that \( L \setminus I \) is not empty. As the input relations of \( Q''_{\text{light}}(\eta) \) contain only light values, \( Q''_{\text{light}}(\eta) \) is skew-free if a share of \( \lambda \) is assigned to each attribute in \( L \setminus I \). By Lemma 5, \( \text{Join}(Q''_{\text{light}}(\eta)) \) can be computed in one round with load \( \tilde{O}(m/\lambda^2) = \tilde{O}(m/p^{1/\rho}) \) using \( \Theta(\lambda|L \setminus I|) \) machines, subject to a certain failure probability \( \delta \). As \( \lambda|L \setminus I| \geq \lambda \) which is a polynomial of \( p \), Lemma 5 allows us to make sure \( \delta \leq 1/p^c \) for any constant \( c \).

By combining the above discussion with Lemma 4, we conclude that \( \text{Join}(Q''(\eta)) \) can be computed in one round with load \( \tilde{O}(m/p^{1/\rho}) \) using \( p''_\eta \) machines, subject to a failure probability at most \( \delta \leq 1/p^c \).

Overall, the load of our algorithm is \( \tilde{O}(p^{1/\rho} + p^2 + m/p^{1/\rho}) \). This brings us to our second main result:

**Theorem 13.** Given a simple binary join query with input size \( m \geq p^3 \) and a fractional edge covering number \( \rho \), we can answer it in the MPC model using \( p \) machines in constant rounds with load \( \tilde{O}(m/p^{1/\rho}) \), subject to a failure probability of at most \( 1/p^c \) where \( c \) can be set to an arbitrarily large constant.

### 7. Concluding Remarks

This paper has introduced an algorithm for computing a natural join over binary relations under the MPC model. Our algorithm performs a constant number of rounds and incurs a load of \( \tilde{O}(m/p^{1/\rho}) \) where \( m \) is the total size of the input relations, \( p \) is the number of machines, and \( \rho \) is the fractional edge covering number of the query. The load matches a known lower bound up to a polylogarithmic factor. Our techniques heavily rely on a new finding, which we refer to as the *isolated cartesian product theorem*, on the join problem’s mathematical structure.

We conclude the paper with several remarks:

- The assumption \( p^3 \leq m \) can be relaxed to \( p \leq m^{1-\epsilon} \) for an arbitrarily small constant \( \epsilon > 0 \). Recall that our algorithm incurs a load of \( \tilde{O}(p^{1/\rho} + p^2 + m/p^{1/\rho}) \) where the terms \( \tilde{O}(p^{1/\rho}) \) and \( \tilde{O}(p^2) \) are both due to the computation of statistics (in preprocessing and Step 2, respectively). In turn, these statistics are needed to allocate machines for subproblems. By using the machine-allocation techniques in [10], we can avoid most of the statistics communication and reduce the load to \( \tilde{O}(p^2 + m/p^{1/\rho}) \).
- Our algorithm description has not touched upon tricks for reducing the number of rounds. A short version [19] of the paper explains how to implement the algorithm in 3 rounds (excluding preprocessing).
- In the external memory (EM) model [4], we have a machine equipped with \( M \) words of internal memory and an unbounded disk that has been formatted into *blocks* of size \( B \) words. An I/O either reads a block of \( B \) words from the disk to the memory, or overwrites a block with \( B \) words in the memory. A join query \( Q \) is considered solved if every tuple
\( u \in \mathcal{Q} \) has been generated in memory at least once. The challenge is to design an algorithm to achieve the purpose with as few I/Os as possible. There exists a reduction [13] that can be used to convert an MPC algorithm to an EM counterpart. As shown in another short version [12] of this paper, applying the reduction on our algorithm gives an EM algorithm that solves \( \mathcal{Q} \) with \( \tilde{O}\left(\frac{m^c}{M^{\rho-1}}\right) \) I/Os, provided that \( M \geq m^c \) for some positive constant \( c < 1 \) that depends on \( \mathcal{Q} \). The I/O complexity can be shown to be optimal up to a polylogarithmic factor using the lower-bound arguments in [11,17]. We suspect that the constraint \( M \geq m^c \) can be removed by adapting the isolated cartesian product theorem to the EM model.

**REFERENCES**
