Tries

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April 9, 2013

In this lecture, we will discuss the following exact matching problem on strings.

Problem

Let S be a set of strings, each of which has a unique integer id. Given a query string q, a query reports:

- the id of q if it exists in S
- nothing otherwise.

Example

Suppose that $S = \{$aaabb, aab, aabaa, aabab, aba, abbb, abbba, abbbb$\}$. Let the ids of these strings be (from left to right) 1, 2, ..., 8, respectively. Given $q =$ aabaa, a query returns id 3, whereas given $q =$ abab, it returns nothing.

Think

How is this problem related to inverted indexes and search engines?

Notations and A Naive Solution

Let

- A be the alphabet (i.e., every character of any string must come from A).
- $|s|$ be the length of a string s, i.e., the number of characters in s.
- $m = |S|$, i.e., the number of strings in S.
- $n =$ the total length of the strings in S, i.e., $n = \sum_{s \in S} |s|$.

When $|A|$ is small and all strings in S are short (e.g., $|s| \leq 10$ for all $s \in S$), the exact matching problem on strings can be reduced to exact matching on integers. For example, consider that each string s represents an English word, and that every s has length at most 10. We can map s to an integer from 0 to $26^{10} - 1$.

Think

Why does the method no longer work if $|A|$ is large or strings can be arbitrarily long?

Next, we will describe another solution based on a data structure called trie. First, let us define the concept of prefix. Let s be a string of length t. We can write its characters (from left to right) as $s[1], s[2], ..., s[t]$, respectively. Then, for any $i \in [1, t]$, the string formed by the sequence $s[1], ..., s[i]$ is called a prefix of s. Specially, an empty string $\emptyset$ is also a prefix of s.

Example

$s =$ aabaa has 6 prefixes: $\emptyset$, a, aa, aab, aaba, and aabaa.

Let S be a set of strings. We say that a string s is a possible prefix of S if s is a prefix of at least one string in S.

A set S of strings is called prefix-free if no string in S is a prefix of any other string in S. Every set of strings can be made prefix-free by appending a special "termination symbol" to each string in S.

Example

Let $S = \{\texttt{aaabb}, \texttt{aab}, \texttt{aabaa}, \texttt{aabab}, \texttt{aba}, \texttt{abbb}, \texttt{abbba}, \texttt{abbbb}\}$. We can convert S to $S' = \{\texttt{aaabb}\perp, \texttt{aab}\perp, \texttt{aabaa}\perp, \texttt{aabab}\perp, \texttt{aba}\perp, \texttt{abbb}\perp, \texttt{abbba}\perp, \texttt{abbbb}\perp\}$, which is prefix-free.

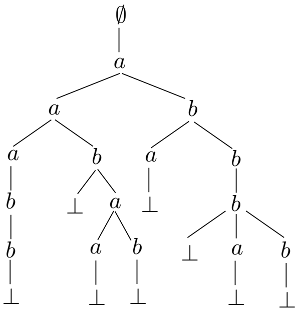
From now on, we will consider that S is prefix-free, and that every string in S ends with $\perp$.

Tries

The trie on S is a tree T defined as follows:

- Each node u of T corresponds to a distinct possible prefix of S. Let $P(u)$ be the prefix that u represents.
- Let u be a node, and v a child node of u. Then:
  - $P(u)$ is a prefix of $P(v)$.
  - $|P(v)| = |P(u)| + 1$.
- Each node u is labeled with a character c, which is the last character of $P(u)$.

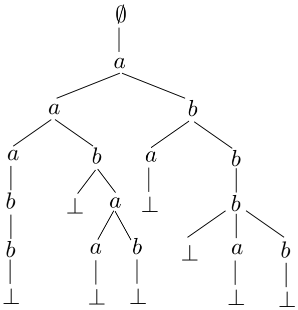
Example: Let $S = \{\texttt{aaabb}\perp, \texttt{aab}\perp, \texttt{aabaa}\perp, \texttt{aabab}\perp, \texttt{aba}\perp, \texttt{abbb}\perp, \texttt{abbba}\perp, \texttt{abbbb}\perp\}$. The trie is:



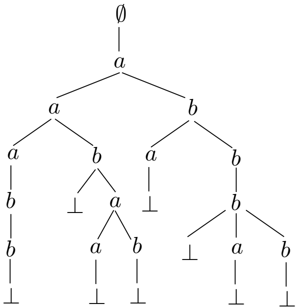
Note that every $\perp$-node u corresponds to a distinct string $s \in S$. We therefore store the id of s at u.

Lemma

The trie on S has at most n nodes.



How do we answer an exact matching query with $q = \texttt{aabaa}$? How about $q = \texttt{abab}$?



How to delete the string aaabb⊥? How about inserting ababb⊥?

Notice that the efficiency of queries, insertions and deletions depends on how well we can solve the following problem:

Given a node u and a character $\sigma \in A \cup \{\perp\}$, how to find the child of v of u that corresponds to σ?

Different tradeoffs exist:

- By organizing the child nodes of u in an array, we can find v in $O(1)$ time, but the array occupies $O(|A|)$ space.
- By organizing the child nodes of u in a binary search tree (BST), we can find v in $O(\log |A|)$ time, and the tree occupies $O(|f|)$ space, where f is the number of child nodes of u.

Theorem

- By using the array implementation, a trie occupies $O(|A|n)$ space, answers a query with string q in $O(|q|)$ time, and supports the insertion and deletion of a string s in $O(|A||s|)$ time.
- By using the BST implementation, a trie occupies $O(n)$ space, answers a query with string q in $O(|q| \log |A|)$ time, and supports the insertion and deletion of a string s in $O(|s| \log |A|)$ time.

Next, we will describe another trie variant, called balanced trie, which occupies $O(n)$ space, and answers a query with string q in $O(\log m + |q|)$ time. The trie, however, is static, namely, it does not support insertions and deletions.

From now on, we consider that S is sorted alphabetically (placing $\perp$ before all characters of A). In general, given a set S' of x sorted strings, we refer to the one in S' whose rank is $\lceil x/2 \rceil$ as the median of S'.

Example

The median of $\{\mathtt{aaabb}\perp, \mathtt{aab}\perp, \mathtt{aabaa}\perp, \mathtt{aabab}\perp, \mathtt{aba}\perp, \mathtt{abbb}\perp, \mathtt{abbba}\perp, \mathtt{abbbb}\perp\}$ is $\mathtt{aabab}\perp$.

Furthermore, given a prefix p, denote by $S(p)$ the set of strings in S with prefix p.

Example

Let $S = \{\mathtt{aaabb}\perp, \mathtt{aab}\perp, \mathtt{aabaa}\perp, \mathtt{aabab}\perp, \mathtt{aba}\perp, \mathtt{abbb}\perp, \mathtt{abbba}\perp, \mathtt{abbbb}\perp\}$. Then $S(\mathtt{aab}) = \{\mathtt{aab}\perp, \mathtt{aabaa}\perp, \mathtt{aabab}\perp\}$.

We also need to define what it means by concatenation. The concatenation of two strings s_1 and s_2 forms a string by appending the characters of s_2 at the end of s_1.

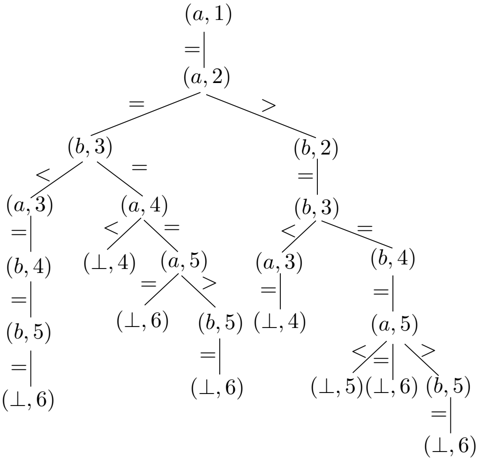
Example

If $s_1 = \texttt{ab}$ and $s_2 = \texttt{bba}$, then concatenation gives $\texttt{abbba}$. If $s_1 = \emptyset$ and $s_2 = \texttt{bba}$, then concatenation gives $\texttt{bba}$. Similarly, if $s_1 = \texttt{ab}$ and $s_2 = \emptyset$, concatenation gives $\texttt{ab}$.

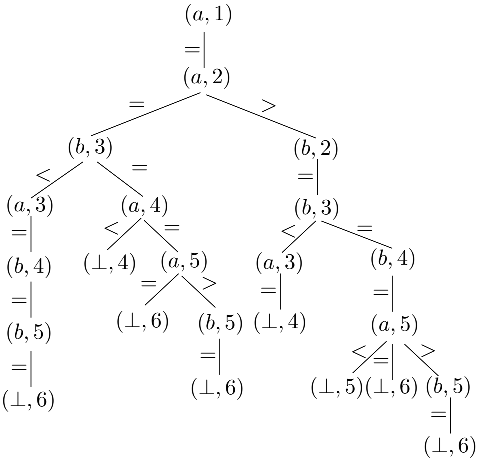
Let S be a set of strings. The balanced trie on S is a tree T defined as follows:

- Every node u in T corresponds to a set $S(u)$ of strings, and carries a label $L(u)$ and a positional index $I(u)$, which will be formally defined below.
- $L(u)$ is the i-th character of the median of $S(u)$, where $i = I(u)$.
- Each u corresponds to a possible prefix $P(u)$ of S, where $P(u)$ is the concatenation of the labels of the nodes on the path from the root to u.
- If u is the root, $S(u) = S$, and $I(u) = 1$.
- u is a leaf if $|S(u)| = 1$ and $I(u) = |s|$, where s is the (only) string in $S(u)$.
- An internal u has at most 3 child nodes $u_<$, $u_=$, and $u_>$ such that:
    - $S(u_<)$ is the set of strings in $S(u)$ alphabetically less than $P(u)$. $I(u_<) = I(u)$.
    - $S(u_=)$ is the set of strings in $S(u)$ that have $P(u)$ as their prefixes. $I(u_=) = I(u) + 1$.
    - $S(u_>)$ is the set of remaining strings in $S(u)$. $I(u_>) = I(u)$

Example: Let $S = \{\texttt{aaabb}\perp, \texttt{aab}\perp, \texttt{aabaa}\perp, \texttt{aabab}\perp, \texttt{aba}\perp, \texttt{abbb}\perp, \texttt{abbba}\perp, \texttt{abbbb}\perp\}$. The balanced trie is:



Each node u is denoted in the form $(L(u), l(u))$.



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A balanced trie occupies $O(n)$ space, and answers a query with string q in $O(\log m + |q|)$ time