INFS 4205/7205: Exercise Set 5

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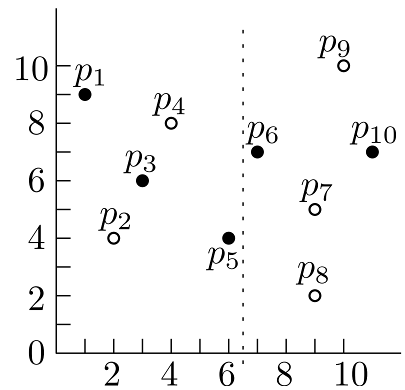
Problem 1. Consider that we run the divide-and-conquer algorithm to solve the dominance screening on the following 2D dataset (i.e., report the white points that are not dominated by the black ones). The algorithm divides the dataset along the x-dimension in halves, where the first half consists of points p_1, p_2, p_3, p_4, p_5, and the second consists of $p_6, p_7, p_8, p_9, p_{10}$. Answer the following questions:

- What is the answer set returned from each half?
- In order to merge the two answer sets into the final answer, the algorithm needs to solve a 1D dominance screening instance. Indicate the 1D points (by specifying their coordinates) in this instance.

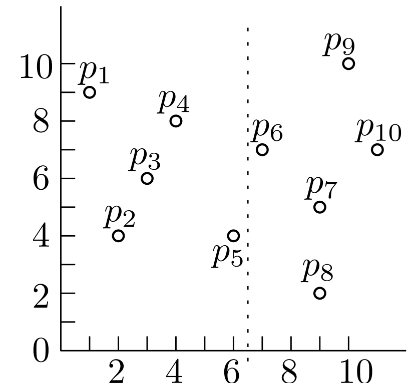


Solution. $\{p_2\}$ is returned from the first half, and $\{p_7, p_8\}$ is returned from the second half.

The 1D dominance screening instance is to report all the white points that are not dominated by any black point, where (i) the 1D black points are $9, 6$ and 4, i.e., the y-coordiantes of points p_1, p_3 and p_5, respectively; (ii) the 1D white points are 5 and 2, i.e., the y-coordiantes of points p_7 and p_8.

Problem 2. Consider that we run the divide-and-conquer algorithm to find the skyline on the following dataset (i.e., report all the points that are not dominated by any other point). The algorithm divides the dataset along the x-dimension in halves, where the first half consists of points p_1, p_2, p_3, p_4, p_5, and the second consists of $p_6, p_7, p_8, p_9, p_{10}$. Answer the following questions:

- What is the answer set returned from each half?
- In order to merge the two answer sets into the final answer, the algorithm needs to solve a 1D dominance screening instance. Indicate the 1D points (by specifying their coordinates) in this instance.



Solution. $\{p_1, p_2\}$ is returned from the first half, and $\{p_6, p_8\}$ is returned from the second half.

The 1D dominance screening instance is to report all the white points that are not dominated by any black point, where (i) the 1D black points are 9 and 4, i.e., the y-coordiantes of points p_1 and p_2; (ii) the 1D white points are 7 and 2, i.e., the y-coordiantes of points p_6 and p_8.

Problem 3. Consider that we run the divide-and-conquer algorithm to find the skyline on the following 3D dataset:

$$p_1 = (1,2,3), p_2 = (2,4,6), p_3 = (4,7,9), p_4 = (4,9,3), p_5 = (6,3,1),$$
$$p_6 = (7,2,9), p_7 = (8,4,5), p_8 = (8,6,7), p_9 = (9,9,8), p_{10} = (9,2,2).$$

The algorithm divides the dataset along the x-dimension in halves, where the first half consists of points p_1, p_2, p_3, p_4, p_5, and the second consists of $p_6, p_7, p_8, p_9, p_{10}$. Answer the following questions:

- What is the answer set returned from each half?
- In order to merge the two answer sets into the final answer, the algorithm needs to solve a 2D dominance screening instance. Indicate the 2D points (by specifying their coordinates) in this instance.

Solution. $\{p_1, p_5\}$ is returned from the first half, and $\{p_6, p_7, p_{10}\}$ is returned from the second half.

The 2D dominance screening instance is to report all the white points that are not dominated by any black point, where (i) the 2D black points are $(2,3)$ and $(3,1)$, i.e., the projections on the yz-plane of points p_1 and p_5; (ii) the 2D white points are $(2,9)$, $(4,5)$ and $(2,2)$, i.e., the projections on the yz-plane of points p_6, p_7 and p_{10}, respectively.

Problem 4. Consider the 2D dominance screening problem again. Recall that we have two sets P, Q of points in $\mathbb{R}^2$. The goal is to identify all the points $q \in Q$ such that q is not dominated by any point in P. Suppose that P and Q have already been sorted by x-dimension. Give an algorithm to solve the problem in $O(n)$ time where $n = |P| + |Q|$. You may assume that all the points in $P \cup Q$ have distinct x-coordinates.

Solution. We scan the points of P and Q in ascending order of x-coordinate. In the meantime, maintain y_{min} as the smallest y-coordinate of the points from P seen so far; $y_{min} = \infty$ at the beginning. We process the next point s as follows:

- If s comes from P, update y_{min} if it is larger than the y-coordinate of s.

- If s comes from Q, report s if its y-coordinate is smaller than y_{min}.

It is easy to verify that the algorithm runs in $O(n)$ time.

Problem 5*. Consider the 3D dominance screening (DS) problem. Recall that we have two sets P, Q of points in $\mathbb{R}^3$. The goal is to identify all the points $q \in Q$ such that q is not dominated by any point in P. Give an algorithm to solve the problem in $O(n \log n)$ time where $n = |P| + |Q|$. Again, you may assume that all the points in $P \cup Q$ have distinct x-coordinates.

Solution. From P, produce two sorted lists: (i) the *x-list*, where the points of P are sorted by x-coordinate, and (ii) the *y-list*, where the points are sorted by y-coordinate. In other words, each point of P is stored twice, once in each list. Do the same from Q. This can be done in $O(n \log n)$ time.

Our algorithm runs in almost the same way as described in the class, except that we divide the input sets in a more careful manner to preserve the point ordering:

- Divide the points of P and Q along the x-dimension in $O(n)$ time as follows:
  - Pick the median x-coordinate value x_{median} among all the points in $P \cup Q$.
  - Divide the points of P into P_1 and P_2, such that P_1 contains all the points with x-coordinates at most x_{median}, and P_2 the rest. We need to produce an x-list and a y-list for P_1, and the same also for P_2. The x-lists of P_1 and P_2 can be obtained simply by splitting the x-list of P at the middle. To obtain their y-lists, we scan the y-list of P; for each point p encountered, append it to the y-list of either P_1 or P_2, depending on the comparison of the x-coordinate of p and x_{median}.
  - In the same manner, divide the points of Q into Q_1 and Q_2, and produce their x- and y-lists, respectively.

- Solve recursively the left DS instance with input sets P_1 and Q_1, and also the right instance with input sets P_2 and Q_2. Let A_{left} and A_{right} be their answer sets. We place the *recursion requirement* that both A_{left} and A_{right} be sorted by y.

- Construct a 2D DS instance of P' and Q', where P' (Q', resp.) is obtained by projecting all the points of P_1 (A_{right}, resp.) onto the yz-space. As all the points in P' and Q' are sorted by y-dimension, the algorithm in Problem 4 solves the problem in $O(n)$ time.

- Combine A_{left} and A' into a set A sorted by y in order to fulfill the recursion requirement. As both A_{left} and A' are sorted by y, this takes $O(n)$ time.

- Return A.

Let $f(n)$ be the time of our algorithm on n points. It holds that $f(n) \le 2 \cdot f(n/2) + O(n)$ and $f(1) = O(1)$. Thus, $f(n) = O(n \log n)$.

Problem 6. Give an algorithm to find the skyline of n points in $\mathbb{R}^d$ (where $d \ge 3$) in $O(n \log^{d-2} n)$ time.

Solution. Plugging in the algorithm in Problem 5 into the analysis discussed in the class, we know that the d-dimension dominance screening problem can be solved in $O(n \log^{d-2} n)$ time for $d \ge 3$.

Then, plug in this result into the skyline analysis discussed in the class, we know that the d-dimension skyline problem can be solved in $O(n \log^{d-2} n)$ time for $d \ge 3$.