Exercises on “the Growth of Functions”

Junhao Gan

ITEE
University of Queensland
Introduction

Last week, we have learned two different ways to decide whether one function $f(n)$ grows faster than another $g(n)$:

- The first one achieves the purpose by finding appropriate “constants $c_1, c_2$”.
- The second is by inspecting the ratio $\frac{f(n)}{g(n)}$ as $n \to \infty$.

In this tutorial, we will apply both methods through some exercises.
Exercise 1

Let $f(n) = 10n + 5$ and $g(n) = n^2$. Prove $f(n) = O(g(n))$ and $g(n) \neq O(f(n))$. 
Exercise 1

Let \( f(n) = 10n + 5 \) and \( g(n) = n^2 \). Prove \( f(n) = O(g(n)) \) and \( g(n) \neq O(f(n)) \).

Direction 1: Constant Finding

\[ f(n) = O(g(n)), \text{ if there exist two constants } c_1 \text{ and } c_2 \text{ such that } f(n) \leq c_1 \cdot g(n) \text{ holds for all } n \geq c_2. \]
Exercise 1

Let \( f(n) = 10n + 5 \) and \( g(n) = n^2 \). Prove \( f(n) = O(g(n)) \) and \( g(n) \neq O(f(n)) \).

Direction 1: Constant Finding

Proof of \( f(n) = O(g(n)) \)

Our mission is to find \( c_1, c_2 \) to make \( f(n) \leq c_1 \cdot g(n) \) hold for all \( n \geq c_2 \). Remember: we do not need to find the smallest \( c_1, c_2 \); instead, it suffices to obtain any \( c_1, c_2 \) that can do the job. Indeed, we will often go for some “easy” selections that can simplify derivation.
Exercise 1

Let \( f(n) = 10n + 5 \) and \( g(n) = n^2 \). Prove \( f(n) = O(g(n)) \) and \( g(n) \neq O(f(n)) \).

Direction 1: Constant Finding

Proof of \( f(n) = O(g(n)) \)

Setting \( c_1 = 10 \), we want:

\[
10n + 5 \leq 10 \cdot n^2
\]

\[
\iff 5 \leq 10n(n - 1)
\]

\[
\iff 5 \leq 10n \quad \text{(for } n \geq 2)\]

\[
\iff 1/2 \leq n
\]

Hence, it suffices to set \( c_2 = 2 \).
Exercise 1

Let \( f(n) = 10n + 5 \) and \( g(n) = n^2 \). Prove \( f(n) = O(g(n)) \) and \( g(n) \neq O(f(n)) \).

Direction 1: Constant Finding

Proof of \( g(n) \neq O(f(n)) \)

Let us prove this by contradiction. Suppose, on the contrary, that \( g(n) = O(f(n)) \). This means the existence of constants \( c_1, c_2 \) such that, we have for all \( n \geq c_2 \)

\[
    n^2 \leq c_1 \cdot (10n + 5)
\]

\[
    \Rightarrow \quad n^2 \leq c_1 \cdot 20n
\]

\[
    \Leftrightarrow \quad n \leq 20c_1
\]

which cannot always hold for all \( n \geq c_2 \). This completes the proof.
Exercise 1

Let \( f(n) = 10n + 5 \) and \( g(n) = n^2 \). Prove \( f(n) = O(g(n)) \) and \( g(n) \neq O(f(n)) \).

Direction 2: Inspecting \( \lim_{n \to \infty} \frac{f(n)}{g(n)} \)

Proof of \( f(n) = O(g(n)) \)

\[
\lim_{n \to \infty} \frac{10n + 5}{n^2} = \lim_{n \to \infty} \frac{10 + 5/n}{n} = 0.
\]

Hence, \( f(n) = O(g(n)) \).
Exercise 1

Let $f(n) = 10n + 5$ and $g(n) = n^2$. Prove $f(n) = O(g(n))$ and $g(n) \neq O(f(n))$.

Direction 2: Inspecting $\lim_{n \to \infty} \frac{f(n)}{g(n)}$

Proof of $g(n) \neq O(f(n))$

$$\lim_{n \to \infty} \frac{n^2}{10n + 5} = \infty.$$ 

Hence, $g(n) \neq O(f(n))$. 
Exercise 2

Let $f(n) = 5 \log_2 n$ and $g(n) = \sqrt{n}$. Prove $f(n) = O(g(n))$ and $g(n) \neq O(f(n))$. 
Direction 1: Constant Finding

Proof of $f(n) = O(g(n))$

Setting $c_1 = 5$, we want:

\[ 5 \log_2 n \leq 5 \cdot \sqrt{n} \]

\[ \iff \log_2 n \leq \sqrt{n} \]

Hence, it suffices to set $c_2 = 64$. 
Direction 1: Constant Finding

Proof of $g(n) \neq O(f(n))$

We prove this by contradiction. Suppose that $g(n) = O(f(n))$. It implies that there exist constants $c_1, c_2$ such that for all $n \geq c_2$, we have

$$\sqrt{n} \leq c_1 \cdot 5 \cdot \log_2 n$$

$$\Leftrightarrow \frac{\sqrt{n}}{\log_2 n} \leq 5c_1$$

which cannot always hold for all $n \geq c_2$. This completes the proof.
Direction 2: Inspecting \( \lim_{n \to \infty} \frac{f(n)}{g(n)} \)

**Proof of** \( f(n) = O(g(n)) \)

\[
\lim_{n \to \infty} \frac{f(n)}{g(n)} = \lim_{n \to \infty} \frac{5 \log_2 n}{\sqrt{n}} = 0.
\]

Thus, we have \( f(n) = O(g(n)) \).

**Proof of** \( g(n) \neq O(f(n)) \).

\[
\lim_{n \to \infty} \frac{g(n)}{f(n)} = \lim_{n \to \infty} \frac{\sqrt{n}}{5 \log_2 n} = \infty.
\]

Hence, \( g(n) \neq O(f(n)) \).
Exercise 3

Given that $10n + 5 = O(n^2)$ and $5 \log_2 n = O(\sqrt{n})$, prove $10n + 5 + 5 \log_2 n = O(n^2 + \sqrt{n})$. 
Direction 1: Constant Finding

Since \(10n + 5 = O(n^2)\) implies the existence of constants \(c_1\) and \(c_2\) such that \(10n + 5 \leq c_1 \cdot n^2\) holds for all \(n \geq c_2\).

Similarly, \(5 \log_2 n = O(\sqrt{n})\) means there exist two constants \(c'_1\) and \(c'_2\) which make \(5 \log_2 n \leq c'_1 \cdot \sqrt{n}\) hold for all \(n \geq c'_2\).

Thus:

\[
10n + 5 + 5 \log_2 n \leq c_1 n^2 + c'_1 \sqrt{n} \leq \max\{c_1, c'_1\} \cdot (n^2 + \sqrt{n})
\]

holds for all \(n \geq \max\{c_2, c'_2\}\).

Therefore, \(10n + 5 + 5 \log_2 n = O(n^2 + \sqrt{n})\).
Direction 2: Inspecting \( \lim_{n \to \infty} \frac{f(n)}{g(n)} \)

Since \( 10n + 5 = O(n^2) \), we have \( \lim_{n \to \infty} \frac{10n+5}{n^2} = c \), where \( c \) is some constant.

Similarly, \( 5 \log_2 n = O(\sqrt{n}) \) indicates that \( \lim_{n \to \infty} \frac{5 \log_2 n}{\sqrt{n}} = c' \), where \( c' \) is some constant.

Both of the above imply that:

\[
\lim_{n \to \infty} \frac{10n + 5 + 5 \log_2 n}{n^2 + \sqrt{n}} = \lim_{n \to \infty} \frac{10n + 5}{n^2 + \sqrt{n}} + \lim_{n \to \infty} \frac{5 \log_2 n}{n^2 + \sqrt{n}} \\
\leq \lim_{n \to \infty} \frac{10n + 5}{n^2} + \lim_{n \to \infty} \frac{5 \log_2 n}{\sqrt{n}} \\
= c + c'.
\]

Therefore, \( 10n + 5 + 5 \log_2 n = O(n^2 + \sqrt{n}) \).
Exercise 4

Consider functions of $n$: $f_1(n)$, $f_2(n)$, $g_1(n)$ and $g_2(n)$ such that:

$$f_1(n) = O(g_1(n)) \text{ and } f_2(n) = O(g_2(n))$$

Prove $f_1(n) + f_2(n) = O(g_1(n) + g_2(n))$. 
Direction 1: Constant Finding

Since $f_1(n) = O(g_1(n))$, there exist constants $c_1$ and $c_2$ such that $f_1(n) \leq c_1 \cdot g_1(n)$ holds for all $n \geq c_2$.

Similarly, $f_2(n) = O(g_2(n))$ implies the existence of constants $c'_1$ and $c'_2$ such that $f_2(n) \leq c'_1 \cdot g_2(n)$ holds for all $n \geq c'_2$.

Thus:

$$f_1(n) + f_2(n) \leq c_1 \cdot g_1(n) + c'_1 \cdot g_2(n) \leq \max\{c_1, c'_1\} \cdot (g_1(n) + g_2(n))$$

for all $n \geq \max\{c_2, c'_2\}$.

Therefore, $f_1(n) + f_2(n) = O(g_1(n) + g_2(n))$. 
Direction 2: Inspecting $\lim_{n \to \infty} \frac{f(n)}{g(n)}$

Since $f_1(n) = O(g_1(n))$, we have $\lim_{n \to \infty} \frac{f_1(n)}{g_1(n)} = c$ for some constant $c$.

Similarly, $f_2(n) = O(g_2(n))$ indicates $\lim_{n \to \infty} \frac{f_2(n)}{g_2(n)} = c'$ for some constant $c'$.

This leads to:

$$\lim_{n \to \infty} \frac{f_1(n) + f_2(n)}{g_1(n) + g_2(n)} = \lim_{n \to \infty} \frac{f_1(n)}{g_1(n) + g_2(n)} + \lim_{n \to \infty} \frac{f_2(n)}{g_1(n) + g_2(n)}$$

$$\leq \lim_{n \to \infty} \frac{f_1(n)}{g_1(n)} + \lim_{n \to \infty} \frac{f_2(n)}{g_2(n)}$$

$$\leq c + c'.$$

Therefore, $f_1(n) + f_2(n) = O(g_1(n) + g_2(n))$. 